

ON LARGE INDECOMPOSABLE BANACH SPACES

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ABSTRACT. Hereditarily indecomposable Banach spaces may have density at most continuum (Plichko-Yost, Argyros-Tolias). In this paper we show that this cannot be proved for indecomposable Banach spaces. We provide the first example of an indecomposable Banach space of density 2^{2^ω} . The space exists consistently, is of the form $C(K)$ and it has few operators in the sense that any bounded linear operator $T : C(K) \rightarrow C(K)$ satisfies $T(f) = gf + S(f)$ for every $f \in C(K)$, where $g \in C(K)$ and $S : C(K) \rightarrow C(K)$ is weakly compact (strictly singular).

1. INTRODUCTION

We say that an infinite dimensional Banach space X is indecomposable (I) if whenever $X = A \oplus B$, then either A or B is finite dimensional. First indecomposable Banach spaces constructed by Gowers and Maurey were hereditarily indecomposable (HI) (see [6], [14]). That is, all their infinite dimensional subspaces were indecomposable. These spaces were separable, but many nonseparable constructions followed e.g., [1]. However on every HI space there is an injective operator into l_∞ , and so HI spaces have density at most 2^ω (see [19]).

Different indecomposable Banach spaces were constructed in [10] and in several papers that followed like [18], [4], [20]. These spaces are of the form $C(K)$ and so they have many decomposable subspaces, actually they must have all separable Banach spaces as subspaces, and cannot be separable. Most of these constructions are of density 2^ω with the exception of [4] which has consistently a smaller density.

In this paper we show that indecomposable Banach spaces of the form $C(K)$ may have density bigger than 2^ω which provides the first example of such Banach spaces. Our construction is not absolute, that is, we prove that it is consistent that such spaces exist. Constructions of Banach spaces which essentially depend on additional set-theoretic axioms or methods are not uncommon among nonseparable spaces. Probably the most known examples are of Shelah from [23] and the so called Kunen space (see [16]) where despite its nonseparability we do not have uncountable biorthogonal systems. Newer important examples include generic Banach spaces of Lopez-Abad and Todorćević ([15]) or Banach spaces without support sets of [11]. The necessity of these additional combinatorial methods in some of the above mentioned results is shown in [25].

The main result of this paper is:

Theorem 1.1. *It is consistent with $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ that there is a compact Hausdorff space L such that:*

- (1) *The density of $C(L)$ is $2^{2^\omega} = \omega_2 > \omega_1 = 2^\omega$,*

The author was partially supported by Polish Ministry of Science and Higher Education research grant N N201 386234.

- (2) *Every linear bounded operator $T : C(L) \rightarrow C(L)$ satisfies $T(f) = gf + S(f)$ for every $f \in C(L)$ where $g \in C(L)$ and S is weakly compact (strictly singular),*
- (3) *$C(L)$ is an indecomposable Banach space, in particular it has no infinite dimensional complemented subspaces of density less or equal to 2^ω ,*
- (4) *$C(L)$ is not isomorphic to any of its proper subspaces nor any of its proper quotients, in particular it is not isomorphic to its hyperplanes.*

This is an immediate consequence of 8.2. We do not know if the existence of such spaces may be proved in ZFC or if this an undecidable problem.

Many other basic related questions remain open. For example, whether there is any bound of the densities of indecomposable Banach spaces (a question due to S. Argyros), however one can show that there is a bound on the densities of Banach spaces with the properties that we obtain.

Our construction is related to the construction from [9] of a $C(K)$ satisfying the first two items of the above in 1.1. The space K of [9] is totally disconnected and so has nontrivial projections all of which can be characterized as being strictly singular perturbations of multiplications by a characteristic functions of clopen sets. The modification however is very indirect, as standard amalgamations of approximating subspaces from [9] do not preserve the connectedness. Actually quite complicated partial order which is used here to force the compact L is designed to take care of the connectedness of the amalgamation. In a sense, what is required is that the metrizable approximations to the final L must predict all possible future amalgamations. So, much more complicated structure of these approximations is required in the present paper than in [9].

It would be very interesting to have a more direct argument which would give the connectedness of the amalgamation. We also feel that the right language for such constructions should be that of Banach algebras, that is, one should rather construct a $C(L)$ than L . Then the connectedness of L is, of course, equivalent the nonexistence of nontrivial idempotent elements in the algebra $C(L)$ which in this context (of few operators) gives the nonexistence of nontrivial (in more relaxed sense of finite dimensional perturbation) idempotents in the algebra $\mathcal{B}(C(L))$ of operators on $C(K)$. However the question of the existence of nontrivial idempotents in an algebra given by its generators could be a difficult question (e.g., [17], [5]) which requires a complicated machinery.

Our approach has this flavour of considering the algebra $C(L)$ only partially. Namely, we take an order complete Banach algebra $C(K)$ for K extremely disconnected and carefully choose a subfamily $\mathcal{F} \subseteq C(K)$ consisting of functions whose ranges are included in $[0, 1]$ which generates the $C(L)$ that is $L = (\Pi\mathcal{F})[K]$ where $\Pi\mathcal{F} : K \rightarrow [0, 1]^\mathcal{F}$ is given by $(\Pi\mathcal{F})(x)(f) = f(x)$ for any $f \in \mathcal{F}$. However most of our arguments are done on the level of underlying compact spaces and are, at first sight, quite detached from the Banach space theoretic structure of the induced function spaces.

Section 1 mainly deals with the extremely disconnected compact K and the spaces of the form $(\Pi\mathcal{F})[K]$ for $\mathcal{F} \subseteq C(K)$. In Section 2 we analyze an auxiliary partial order of some countable subsets of $C(K)$ which could approximate our final \mathcal{F} which would determine L as above. Though the final space L has no nontrivial continuous mappings into itself, the approximating spaces have many homeomorphism which help us in amalgamations of the approximations into better

approximations. Section 4 is devoted to the analysis of the final partial order of approximations which incorporates the auxiliary one from the previous section. In the next Section 5 we consider adding suprema of sequences of functions to our approximating families $\mathcal{F} \subseteq C(K)$. This section contains results that generalize the techniques of Section 4 of [10]. Section 6 is devoted to the main extension lemma, where an approximation to the final $\mathcal{F} \subseteq C(K)$ can be enriched by a supremum of a sequence of pairwise disjoint functions. The last two sections 7 and 8 deal with the final realization of the construction. Only in these two last sections some knowledge of consistency proofs by forcing is required from the reader.

It is worthy to mention that our L is strongly rigid compact space, that is its only continuous mappings are constant functions or the identity (see 5.4. of [21]). A first compact strongly rigid space was obtained by H. Cook ([2]) and V. Trnkova proved that there are no bounds on the weight nor size of such spaces ([24]) in ZFC. However, $C(K)$ s for the compact spaces of [2] or [24] are decomposable. On the other hand there is a necessary and sufficient condition for $C(K)$ to have few operators as above in terms of some rigidity of the dual ball of $C(K)$ (see Theorem 23 of [12]). As in the case of the construction from [9] we conjecture that the space of this paper can be obtained directly from a combinatorial principle called Velleman's simplified morass ([26]) instead of a forcing argument.

We would like to thank Rogério Fajardo for countless hours of discussions back in the years 2005-07 concerning the problem of large indecomposable spaces, to Roman Pol for his remarks related to strongly rigid compact spaces and to Adam Skalski for his remarks on idempotent elements in C^* -algebras.

Most unexplained set-theoretic and logical concepts can be found in [13] or [7], topological terminology is based on [3] and the basics on $C(K)$ Banach spaces can be found in [22].

2. CONTINUOUS IMAGES OF SOME EXTREMELY DISCONNECTED COMPACT SPACE

Let $A, A' \subseteq \omega_2$. We say that $A < A'$ if each element of A is smaller in the ordinal order on ω_2 than all elements of A' . Consider $Fr(A)$, the free Boolean algebra generated by independent family $(a_\xi : \xi \in A)$ and its completion $Co(A)$. It is well known that $Fr(A)$ and so $Co(A)$ satisfy the c.c.c., so every element of $Co(A)$ can be considered as a supremum of an antichain from $Fr(A)$.

If $A \subseteq B$, then there is a natural embedding $i_{BA} : Co(A) \rightarrow Co(B)$ which induces, by the Stone duality a continuous surjection $\rho_{AB} : K_B \rightarrow K_A$, where K_A and K_B are the Stone spaces of $Co(A)$ and $Co(B)$ respectively. We put $K = K_{\omega_2}$. If L is a compact Hausdorff space, then $C(L)$ denotes the Banach space of all real-valued continuous functions on L with the supremum norm. $C_I(K)$ will denote the set of all elements of $C(K)$ whose ranges are included in $[0, 1]$.

In general if \mathcal{A} is a Boolean algebra, then $S(\mathcal{A})$ denotes its Stone space, $[a]$ will denote the basic clopen set of the Stone space of a Boolean algebra where a belongs. So, in particular $K_A = S(Co(A))$. We will use the notation $S(Co(A))$ when we want to exploit the fact that its points are subsets of $Co(A)$ (ultrafilters).

Suppose $\alpha \in \omega_2$. Of course the Stone space of $Fr([\alpha, \alpha + \omega))$ is homeomorphic with the Cantor set 2^ω . By the standard surjection from 2^ω onto $[0, 1]$ we mean the function $f(x) = \sum_{n \in \mathbb{N}} \frac{x(i)}{2^i}$ which is irreducible. $d_\alpha : K \rightarrow [0, 1]$ will denote the function obtained in similar way from $S(Fr([\alpha, \alpha + \omega)))$ instead of 2^ω :

Definition 2.1. Suppose $\alpha \in \omega_2$. $d_\alpha : K \rightarrow [0, 1]$ is defined by

$$d_\alpha(x) = \sum_{n \in \mathbb{N}} \frac{t_x(i)}{2^i},$$

where $t_x(i) = 1$ if $a_{\alpha+i} \in x$ and $t_x(i) = 0$ if $a_{\alpha+i} \notin x$.

Definition 2.2. Suppose that $A, B \subseteq \omega_2$. $Co(A) \otimes Co(B)$ is the subalgebra of $Co(A \cup B)$ generated by $Co(A) \cup Co(B)$.

Lemma 2.3. Let $A, B \subseteq \omega_2$. Let $u \in S(Co(A))$ and $v \in S(Co(B))$ be such that $u \cap Co(A \cap B) = v \cap Co(A \cap B)$. Then there is a unique ultrafilter w in the Stone space of $Co(A) \otimes Co(B)$ such that $u, v \subseteq w$.

Proof. If $u \cup v$ does not have the finite intersection property, there are disjoint $\Sigma(a_i \cap b_i) \in u$ and $\Sigma(a'_i \cap c_i) \in v$ such that $a_i, a'_i \in Fr(A \cap B)$, $b_i \in Fr(A \setminus B)$ and $c_i \in Fr(B \setminus A)$. Hence $\Sigma a_i \in u \cap Co(A \cap B)$ and $\Sigma a'_i \in v \cap Co(A \cap B)$ as they are bigger elements. The hypothesis then, gives that $a_i \cap a'_j \neq 0$ for some i, j . But then $a_i \cap a'_j \cap b_i \cap c_j \neq 0$ by the independence of all elements of $Fr(A \cap B)$, $Fr(A \setminus B)$, $Fr(B \setminus A)$ which contradicts the disjointness of the original elements. \square

Lemma 2.4. Let $A, B \subseteq \omega_2$. Let $u \in S(Co(A))$ and $v \in S(Co(B))$ be such that $u \cap Co(A \cap B) = v \cap Co(A \cap B)$. Then there are ultrafilters $w \in K_{A \cup B}$ and $w' \in K$ such that $u, v \subseteq w, w'$.

Proof. Extend the ultrafilter from 2.3. \square

Corollary 2.5. Suppose $A, B \subseteq \omega_2$ are disjoint and $u \in K_A$ and $v \in K_B$. Then there are ultrafilters $w \in K_{A \cup B}$ and $w' \in K$ such that $u, v \subseteq w, w'$.

Definition 2.6. If $F \subseteq C_I(K)$, then by $\Pi F : K \rightarrow [0, 1]^F$ we mean the function defined by

$$(\Pi F(x))(f) = f(x),$$

for each $x \in K$ and each $f \in F$. The image of ΠF in $[0, 1]^F$ is denoted by ∇F . If $\mathcal{G} \subseteq \mathcal{F}$, then $\pi_{\mathcal{G}\mathcal{F}}$ denotes the projection from $\nabla \mathcal{F}$ onto $\nabla \mathcal{G}$.

Lemma 2.7. Suppose that $\mathcal{F} \subseteq C_I(K)$. $\nabla \mathcal{F}$ is connected if and only if $\nabla\{f_1, \dots, f_m\}$ is connected for any finite $f_1, \dots, f_m \in \mathcal{F}$.

Proof. $\nabla\{f_1, \dots, f_m\}$ is a continuous image of $\nabla \mathcal{F}$ by taking the canonical projections onto the coordinates indexed by f_1, \dots, f_m hence the connectedness is preserved.

Now suppose $A, B \subseteq \nabla \mathcal{F}$ are nonempty complementary clopen sets. Any open set can be covered by basic open sets determined by finitely many coordinates. But A, B are clopen, hence compact, so such covers have finite subcovers. It follows that

$$A = \bigcup_{i \leq n} \bigcap_{j \leq k} (\Pi \mathcal{F})[f_{i,j}^{-1}[U_{i,j}]], \quad B = \bigcup_{i' \leq n'} \bigcap_{j' \leq k'} (\Pi \mathcal{F})[g_{i',j'}^{-1}[V_{i',j'}]],$$

where $\{f_{i,j}, g_{i',j'} : i \leq n, j \leq k, i' \leq n', j' \leq k'\} \subseteq \mathcal{F}$ and $U_{i,j}, V_{i',j'}$ are open subsets of $[0, 1]$. However this means that already $\nabla\{f_{i,j}, g_{i',j'} : i \leq n, j \leq k, i' \leq n', j' \leq k'\}$ is not connected. \square

Lemma 2.8. Suppose $A' \subseteq \omega_2$ consists only of limit ordinals. Then $\nabla\{d_\alpha : \alpha \in A'\}$ is homeomorphic to $[0, 1]^{A'}$.

Proof. As the sets $[\alpha, \alpha + \omega)$ for $\alpha \in A'$ are pairwise disjoint, using 2.4, for any $(x_\alpha)_{\alpha \in A'} \in \Pi_{\alpha \in A'} S(Fr([\alpha, \alpha + \omega))$ there is $x \in S(Fr(\bigcup_{\alpha \in A'} [\alpha, \alpha + \omega)))$ such that $x \cap Fr([\alpha, \alpha + \omega)) = x_\alpha$. So $\Pi\{d_\alpha : \alpha \in A'\}$ is onto $[0, 1]^{\{d_\alpha : \alpha \in A'\}}$ which is of course homeomorphic to $[0, 1]^{A'}$, hence its image $\nabla\{d_\alpha : \alpha \in A'\}$ is homeomorphic to $[0, 1]^{A'}$ as required. \square

Definition 2.9. $f : K \rightarrow \mathbb{R}$ is said to depend on a Boolean algebra $\mathcal{A} \subseteq Co(\omega_2)$ if and only if whenever $x \cap \mathcal{A} = y \cap \mathcal{A}$, then $f(x) = f(y)$. If $A \subseteq \omega_2$ we say that f depends on A if it depends on $Co(A)$.

Lemma 2.10. Each $f \in C(K)$ depends on a countable set $A \subseteq \omega_2$.

Proof. For any rationals $a < b < c < d$ consider a closed $F_{b,c} = f^{-1}[[b, c]]$ and an open $U_{a,d} = f^{-1}[(a, d)]$. Clearly $F_{b,c} \subseteq U_{a,d}$. Let $\mathcal{A} \subseteq Co(\omega_2)$ be a countable subalgebra such that for any rationals $a < b < c < d$ there is a clopen $U \in \mathcal{A}$ such that

$$F_{a,b} \subseteq U \subseteq U_{a,b}.$$

It is enough to show that if $u \cap \mathcal{A} = v \cap \mathcal{A}$, then $f(v) = f(u)$. If not, then there are rational $a < b < c < d$ such that $u \in F_{b,c}$ and $v \notin U_{a,d}$, hence there is $U \in \mathcal{A}$ such that $u \in U$ and $v \notin U$. Now let $A \subseteq \omega_2$ be a countable set such that $\mathcal{A} \subseteq Co(A)$, it exists since $Co(\omega_2)$ is c.c.c. \square

Lemma 2.11. Suppose that all functions from $\mathcal{F} \subseteq C_I(K)$ depend on $A \subseteq \omega_2$ and that $A \cap [\alpha, \alpha + \omega) = \emptyset$. Then $\nabla(\mathcal{F} \cup \{d_\alpha\}) = (\nabla\mathcal{F}) \times [0, 1]$.

Proof. Let $t \in \nabla\mathcal{F}$ and $x \in [0, 1]$ and let $u', v' \in Co(\omega_2)$ be such that $(\Pi\mathcal{F})(u') = t$ and $d_\alpha(v') = x$. Put $v = v' \cap Co([\alpha, \alpha + \omega))$ and $u = u' \cap Co(A)$. Use 2.5 to obtain $w \in Co(\kappa)$ with $w \cap Co([\alpha, \alpha + \omega)) = v' \cap Co([\alpha, \alpha + \omega))$ and $w \cap Co(A) = u' \cap Co(A)$. We have that $(\Pi\mathcal{F})(w) = t$ and $d_\alpha(w) = x$ and so $\Pi(\mathcal{F} \cup \{d_\alpha\})(w) = (t, x)$. \square

Lemma 2.12. If $A \subseteq \omega_2$ is countable then $C(K_A)$ has the cardinality of the continuum. In particular the cardinality of the family of all functions which depend on a countable A is continuum.

Proof. $Fr(A)$ is dense in $Co(A)$. As $Fr(A)$ has c.c.c., any element of $Co(A)$ is the union of a countable antichain in $Fr(A)$. Hence, the cardinality of $Co(A)$ is continuum. By the Stone-Weierstrass theorem any element of $C(K_A)$ can be approximated by a finite linear combination of characteristic functions of elements from $Co(K)$, so elements of $C(K_A)$ can be associated with countable subsets of $[\mathbb{R} \times Co(A)]^{<\omega}$ which has the size of continuum as required. \square

Proposition 2.13. Let $\sigma : \omega_2 \rightarrow \omega_2$ be a bijection. Then there is a unique isomorphism denoted $h(\sigma)$ of the Boolean algebras $co(Fr(\omega_2))$ and $co(Fr(\omega_2))$ which sends a_ξ to $a_{\sigma(\xi)}$. We say that such an isomorphism is induced by σ . Then there is a unique homeomorphism of K denoted $\phi(\sigma)$ such that $\phi(\sigma)[[a_\xi]] = [a_{\sigma^{-1}(\xi)}]$. We say that such a homeomorphism is induced by σ . If $f \in C_I(K)$ depends on $S_f \subseteq \omega_2$, then $f \circ \phi(\sigma)$ depends on $\sigma[S_f]$. If two bijections σ, σ' agree on S_f , then $f \circ \phi(\sigma) = f \circ \phi(\sigma')$. If $\sigma, \sigma' : \omega_2 \rightarrow \omega_2$ are two bijections, then $\phi(\sigma) \circ \phi(\sigma') = \phi(\sigma' \circ \sigma)$.

Proof. Clearly the bijection uniquely determines an isomorphism of the algebra $Fr(\omega_2)$ which sends a_ξ to $a_{\sigma(\xi)}$. Now note that by the Sikorski extension theorem there is an extension to a homomorphism from $Co(Fr(\omega_2))$ into $Co(Fr(\omega_2))$. By

the density of $Fr(\omega_2)$ in $Co(Fr(\omega_2))$ we get that the extension $h(\sigma)$ must be an isomorphism onto $Co(Fr(\omega_2))$ satisfying

$$h(\sigma)(\{\sup\{b_i : i \in \mathbb{N}\}\}) = \sup\{h(\sigma)(b_i) : i \in \mathbb{N}\}$$

for any $b_i \in Fr(\omega_2)$ and $i \in \mathbb{N}$. It follows that $h(\sigma)[Co(A)] = Co[\sigma[A]]$.

Let $\phi(\sigma)$ denote the dual mapping to $h(\sigma)$ obtained via the Stone duality. It is clear that it is a homeomorphism of the Stone space K of $co(Fr(\omega_2))$. By its definition $\phi(\sigma)(u) = h(\sigma)^{-1}[u]$, so $\phi(\sigma)[[a_\xi]] = \{h(\sigma)^{-1}[u] : u \in [a_\xi]\} = \{h(\sigma)^{-1}[u] : a_\xi \in u\} = \{v : h(\sigma)^{-1}(a_\xi) \in v\} = \{v : a_{\sigma^{-1}(\xi)} \in v\} = [a_{\sigma^{-1}(\xi)}]$ as required.

Now if $x \cap co(\sigma[S_f]) = y \cap co(\sigma[S_f])$, then $x \in [a]$ if and only if $y \in [a]$ for any $a \in co(\sigma[S_f])$, it follows from the above that $\phi(\sigma)(x) \in [b]$ if and only if $\phi(\sigma)(y) \in [b]$ for any $b \in co(S_f)$, hence $f \circ \phi(\sigma)(x) = f \circ \phi(\sigma)(y)$ for such x, y . In other words $f \circ \phi(\sigma)$ depends on $\sigma[S_f]$.

To prove next part of the proposition take any $x \in K$. $h(\sigma)$ agrees with $h(\sigma')$ on $Co(S_f)$ by the uniqueness of the $h(\sigma)$ and $h(\sigma')$. Note that if two functions agree on a set A , then the preimages of sets with respect to them have the same intersections with A . So, if $b \in co(S_f)$, then $\phi(\sigma)(x) \in [b]$ if and only if $b \in h(\sigma)^{-1}[x]$ if and only if $b \in h(\sigma')^{-1}[x]$ if and only if $\phi(\sigma')(x) \in [b]$, in other words $\phi(\sigma)(x) \cap Co(S_f) = \phi(\sigma')(x) \cap Co(S_f)$, hence $f \circ \phi(\sigma)(x) = f \circ \phi(\sigma')(x)$.

To prove the last part of the proposition, take two bijections $\sigma, \sigma' : \omega_2 \rightarrow \omega_2$. Given an $x \in S(Co(\omega_2))$, by the definition $(\phi(\sigma) \circ \phi(\sigma'))(x) = \sigma^{-1}[(\sigma')^{-1}[x]] = \sigma^{-1}[\{a \in Co(\omega_2) : \sigma'(a) \in x\}] = \{b \in Co(\omega_2) : (\sigma' \circ \sigma)(b) \in x\} = (\sigma' \circ \sigma)^{-1}[x] = \phi(\sigma' \circ \sigma)(x)$.

□

Lemma 2.14. *Let $\sigma : \omega_2 \rightarrow \omega_2$ be a bijection and $\{f_1, \dots, f_k\} \subseteq C_I(K)$. Then $\nabla\{f_1, \dots, f_k\} = \nabla\{f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma)\}$.*

Proof. Any point of $\nabla\{f_1, \dots, f_k\}$ is of the form $\Pi\{f_1, \dots, f_k\}(x)$ for some $x \in K$, but this point is also of the form $\Pi\{f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma)\}(\phi(\sigma)^{-1}(x))$ which is in $\nabla\{f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma)\}$. Conversely any point of $\nabla\{f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma)\}$ is of the form $\Pi\{f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma)\}(x)$ for some $x \in K$, but this point is also of the form $\Pi\{f_1, \dots, f_k\}(\phi(\sigma)(x))$ which is in $\nabla\{f_1, \dots, f_k\}$.

□

3. AN AUXILIARY PARTIAL ORDER OF APPROXIMATIONS

Definition 3.1. *Suppose that $\xi < \omega_2$ and $A \subseteq B \subseteq \omega_2$. $\Sigma_{\xi, A}(B)$ is the set of all bijections of ω_2 satisfying*

$$\sigma[A] \cap (A \setminus \xi) = \emptyset, \quad \sigma|(\xi \cup (\omega_2 \setminus B)) = Id_{\xi \cup (\omega_2 \setminus B)}.$$

An ideal of subsets of a set A is a family of subsets of A which contains all finite subsets of A and is closed under taking finite unions and taking subsets of its elements. If \mathcal{J} is a family of sets then $\langle \mathcal{J} \rangle$ is the ideal of subsets of $\bigcup \mathcal{J}$ generated by \mathcal{J} , i.e., the family of all subsets of finite unions of elements of \mathcal{J} . We say that an ideal \mathcal{I} of subsets of A is proper if it does not contain A ; it is countably generated if there is a countable $\mathcal{J} \subseteq \mathcal{I}$ such that $\langle \mathcal{J} \rangle = \mathcal{I}$.

Definition 3.2. *We define a partial order \mathbb{P} which consists of conditions of the form $p = (A_p, \mathcal{F}_p, \mathcal{I}_p)$ such that*

- (1) $A_p \in [\omega_2]^{\leq \omega}$, $[\alpha, \alpha + \omega] \subseteq A_p$ for all $\alpha \in A_p$

- (2) $\{d_\xi : \xi \in A_p\} \subseteq \mathcal{F}_p \in [C_I(K)]^{\leq \omega}$,
- (3) \mathcal{I}_p is a proper, countably generated ideal of subsets of A_p
- (4) Each $f \in \mathcal{F}_p$ depends on a countable subset $S_f \in \mathcal{I}_p$.
- (5) For every $f_1, \dots, f_k \in \mathcal{F}_p$, for every $\xi \in A_p$ and $A \in \mathcal{I}_p$ there is $\sigma \in \Sigma_{\xi, A}(A_p)$, with

$$f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma) \in \mathcal{F}_p.$$

- (6) $\nabla \mathcal{F}$ is a connected compact space.

We let $p \leq q$ if and only if $A_p \supseteq A_q$, $\mathcal{F}_p \supseteq \mathcal{F}_q$ and $\mathcal{I}_p \supseteq \mathcal{I}_q$ for $p, q \in \mathbb{P}$.

Definition 3.3. Let $p, q \in \mathbb{P}$ we say that they are isomorphic if and only if $A_p \cap A_q < A_p \setminus A_q < A_q \setminus A_p$ or $A_p \cap A_q < A_q \setminus A_p < A_p \setminus A_q$ and there is a bijection $\tau : \omega_2 \rightarrow \omega_2$ such that $\tau[A_p] = A_q$ and $\tau|(A_p \cap A_q) = Id_{A_p \cap A_q}$ and

- (1) $\mathcal{F}_q = \{f \circ \phi(\tau) : f \in \mathcal{F}_p\}$,
- (2) $\mathcal{I}_q = \{\tau[A] : A \in \mathcal{I}_p\}$.

Lemma 3.4. Suppose that $p, q \in \mathbb{P}$ are isomorphic. Then $r = (A_p \cup A_q, \mathcal{F}_p \cup \mathcal{F}_q, \langle \mathcal{I}_p \cup \mathcal{I}_q \rangle)$ is a condition of \mathbb{P} and $r \leq p, q$.

Proof. We only need to prove that $r \in \mathbb{P}$. The conditions (1), (2) (4) of 3.2 are clear. To prove (3) we need to note the properness of the ideal $\langle \mathcal{I}_p \cup \mathcal{I}_q \rangle$. Note that given $A \in \mathcal{I}_p$ and $B \in \mathcal{I}_q$, if $A_p \cup A_q \subseteq A \cup B$, then $A_q \subseteq \tau[A] \cup B$ which contradicts the properness of \mathcal{I}_q using (2) of 3.2. Now we will prove (5). Assume that

$$A_p \cap A_q < A_p \setminus A_q < A_q \setminus A_p \neq \emptyset$$

and that $\xi_0 = \min(A_p \setminus A_q)$ and $\eta_0 = \min(A_q \setminus A_p)$. Let $\tau : \omega_2 \rightarrow \omega_2$ be a bijection of ω_2 such that $\tau^2 = \text{id}$, $\tau[A_p] = A_q$ and $\tau \upharpoonright A_p \cap A_q = Id_{A_p \cap A_q}$ which witnesses that p and q are isomorphic, that is, in particular that $\mathcal{F}_q = \{f \circ \phi(\tau) : f \in \mathcal{F}_p\}$ and $\mathcal{I}_q = \{\tau[A] : A \in \mathcal{I}_p\}$. These properties of τ can be easily arranged as $f \circ \phi(\tau) = f \circ \phi(\tau')$ whenever τ and τ' agree on A_r and $f \in \mathcal{F}_r$ by 2.13.

Consider a finite subset of \mathcal{F}_r , a $\xi \in A_r$ and an element of \mathcal{I}_r from (5). We may assume, by adding new elements if necessary that this finite set consists of elements $f_1, \dots, f_m \in \mathcal{F}_p$ and $g_1, \dots, g_k \in \mathcal{F}_q$ for $m, k \in \mathbb{N}$ and that the element of \mathcal{I}_r is of the form $A \cup B$ where $A \in \mathcal{I}_p$ and $B \in \mathcal{I}_q$. We may assume that $S_{f_1} \cup \dots \cup S_{f_m} \subseteq A$ and $S_{g_1} \cup \dots \cup S_{g_k} \subseteq B$ and that $\tau[A] = B$, in particular that $A \cap A_p \cap A_q = B \cap A_p \cap A_q$. We need to find $\sigma \in \Sigma_{\xi, A \cup B}(A_r)$ such that $f_1 \circ \phi(\sigma), \dots, f_m \circ \phi(\sigma), g_1 \circ \phi(\sigma), \dots, g_k \circ \phi(\sigma) \in \mathcal{F}_r$.

Case 1. $\xi \in A_p \cap A_q$.

Let $\sigma_1 \in \Sigma_{\xi_0, A}(A_p)$ be such that $f_1 \circ \phi(\sigma_1), \dots, f_k \circ \phi(\sigma_1) \in \mathcal{F}_p$. Let $\sigma_2 \in \Sigma_{\xi, B}(A_q)$ be such that $g_1 \circ \phi(\sigma_2), \dots, g_k \circ \phi(\sigma_2) \in \mathcal{F}_r$.

Define a function $\sigma' : A \cup B \rightarrow A_p \cup A_q$ so that $\sigma'(\alpha) = \sigma_1(\alpha)$ for $\alpha \in A \setminus B$ and $\sigma'(\alpha) = \sigma_2(\alpha)$ for $\alpha \in B$.

We have that $\sigma'|(B \cap \xi) = \sigma_2|(B \cap \xi) = Id_{B \cap \xi}$. Now calculate:

$$\sigma'[(A \cup B) \setminus \xi] \cap (A \cup B) = [\sigma_1[A \setminus \xi_0] \cap (A \setminus \xi_0)] \cup [\sigma_2[B \setminus \xi] \cap B] = \emptyset.$$

Note also that σ' is an injection since it is a union of two injections whose ranges are disjoint. Finally we will extend σ' to a bijection σ of ω_2 which is in $\Sigma_{\xi, A \cup B}(A_r)$. For this we need the following:

Claim: $(A_q \cup A_p) \setminus (\xi \cup A \cup B)$ and $(A_q \cup A_p) \setminus (\xi \cup \sigma[A \cup B])$ are infinite.

Proof of the claim: Case A. $(A \cup B) \setminus \xi_0$ is finite.

Then, as we assume that $A_p \setminus A_q$ is nonempty and so infinite by 1) of 3.2, we have that $[\xi_0, \xi_0 + \omega) \subseteq A_p \setminus \xi_0$, so both of the differences must have infinite intersections with the above interval

Case B. $(A \cup B) \setminus \xi_0$ is infinite.

As we have $\tau[A] = B$, this means that both $A \setminus \xi_0$ and $B \setminus \xi_0$ are infinite, in particular $A \setminus \xi_0, \sigma'[A \setminus \xi_0] \subseteq A_p \setminus A_q$ are infinite and disjoint. Also we have

$$A \setminus \xi_0 \subseteq (A_q \cup A_p) \setminus (\xi \cup \sigma'[A \cup B]),$$

$$\sigma'[A \setminus \xi_0] \subseteq (A_q \cup A_p) \setminus (\xi \cup A \cup B).$$

So both of the sets from the claim are infinite which completes the proof of the claim.

Now we use the claim to define a bijection σ'' of $(A_p \cup A_q) \setminus \xi$ which extends $\sigma'|((A \cup B) \setminus \xi)$. Finally extend it to a bijection σ of ω_2 by adding $Id_{\omega_2 \setminus [(A_p \cup A_q) \setminus \xi]}$. Note that $\sigma'|((A \cup B) \cap \xi) = Id_{(A \cup B) \cap \xi} = \sigma_1|(B \cap \xi)$ and so σ can be the identity while restricted to $\xi \cup (\omega_2 \setminus A_r)$ resulting in $\sigma \in \Sigma_{\xi, A \cup B}(A_r)$.

Now note that for $i = 1, \dots, m$ we have $f_i \circ \phi(\sigma) = f_i \circ \phi(\sigma_1) \in \mathcal{F}_p \subseteq \mathcal{F}_r$ by 2.13 because $S_{f_i} \subseteq A$ and σ_1 agrees with σ on A . Similarly for $i = 1, \dots, k$ $g_i \circ \phi(\sigma), \dots, g_i \circ \phi(\sigma_1), \mathcal{F}_q \subseteq \mathcal{F}_r$ by 2.13 because $S_{g_i} \subseteq A$ and σ_2 agrees with σ on B . Hence $\sigma \in \Sigma_{\xi, A \cup B}(A_r)$ and $f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma), g_1 \circ \phi(\sigma), \dots, g_k \circ \phi(\sigma) \in \mathcal{F}_r$ as required in 5).

Case 2. $\xi \in A_p \setminus A_q$

Let $\sigma_1 \in \Sigma_{\xi, A}(A_p)$ be such that $f_1 \circ \phi(\sigma_1), \dots, f_m \circ \phi(\sigma_1) \in \mathcal{F}_p$. Let $\sigma_2 \in \Sigma_{\xi_0, B}(A_q)$ be such that $g_1 \circ \phi(\sigma_2), \dots, g_k \circ \phi(\sigma_2) \in \mathcal{F}_q$. As $\sigma_1|_{\xi_0} = Id_{\xi_0} = \sigma_2|_{\xi_0}$ there is a bijection σ of ω_2 such that $\sigma|_{A_r} = \sigma_1|_{A_p} \cup \sigma_2|_{A_q}$ and σ is the identity on the remaining part of ω_2 . Note that $\sigma|_{A_r \cap \xi} = id_\xi$ and that

$$\sigma[A \cup B] \cap (A \cup B) \setminus \xi \subseteq (\sigma_1[A] \cap A \setminus \xi) \cup (\sigma_2[B] \cap B \setminus \xi_0) = \emptyset.$$

Hence $\sigma \in \Sigma_{\xi, A \cup B}(A_r)$. Since f_1, \dots, f_k depend on A_p we have $f_i \circ \phi(\sigma) = f_i \circ \phi(\sigma_1)$ also since g_1, \dots, g_k depend on A_q we have $f_i \circ \phi(\sigma) = f_i \circ \phi(\sigma_2)$, hence $f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma), g_1 \circ \phi(\sigma), \dots, g_k \circ \phi(\sigma) \in \mathcal{F}_r$ which completes the proof of (4) in this case.

Case 3. $\xi \in A_q \setminus A_p$

Let $\sigma \in \Sigma_{\xi, B}$ be such that $g_1 \circ \phi(\sigma), \dots, g_k \circ \phi(\sigma) \in \mathcal{F}_q$. As $\sigma|_\xi = Id_\xi$ and all elements of \mathcal{F}_p depend on $A_p \subseteq \xi$ we have that $f_i \circ \phi(\sigma) = f_i$ hence $f_1 \circ \phi, \dots, f_m \circ \phi, g_1 \circ \phi, \dots, g_k \circ \phi \in \mathcal{F}_r$. Also $(A \cup B) \setminus \xi = A \setminus \xi$ as $B \subseteq A_p \subseteq \xi$, so $\sigma \in \Sigma_{\xi, A \cup B}$ as required.

To prove (6) we will show that for every finite $f_1, \dots, f_m \in \mathcal{F}_r$ there are $f'_1, \dots, f'_m \in \mathcal{F}_q$ such that $\nabla\{f_1, \dots, f_m\}$ is homeomorphic to $\nabla\{f'_1, \dots, f'_m\}$. This will be enough by 2.7 since $\nabla\{f'_1, \dots, f'_m\}$ is a continuous image of $\nabla\mathcal{F}_r$ (by projecting), so is connected by (5) of 3.2 for $q \in \mathbb{P}$.

Let $A = S_{f_1} \cup \dots \cup S_{f_m}$ and $B = S_{f'_1} \cup \dots \cup S_{f'_m}$. Let $\sigma_1 \in \Sigma_{\eta_0, \tau[A]}(A_q)$. Define $\sigma' : A \cup B \rightarrow A_p \cup A_q$ by $\sigma'(\alpha) = \sigma_1(\alpha)$ if $\alpha \in B$ and $\sigma'(\alpha) = \tau(\alpha)$ if $\alpha \in A$. σ is well-defined because both σ_1 and τ are the identity on $A_p \cap A_q$. As $\sigma_1|_{B \setminus A}$ and $\tau|_{A \setminus B}$ have disjoint ranges it follows that σ' is an injection. So σ' can be extended to a bijection σ of ω_2 .

Now note that for $i = 1, \dots, k$ we have $f_i \circ \phi(\tau) = f_i \circ \phi(\sigma)$ by 2.13 as τ and σ agree on $S_{f_i} \subseteq A$, so using (2) of 3.3 we have $f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma), g_1 \circ \phi(\sigma), \dots, g_k \circ \phi(\sigma) \in \mathcal{F}_q$. \square

Lemma 3.5. *Suppose that $(p_n)_{n \in \mathbb{N}}$ is a sequence of conditions of \mathbb{P} satisfying $p_{n+1} \leq p_n$ and $p_n = (A_n, \mathcal{F}_n, \mathcal{I}_n)$ for each $n \in \mathbb{N}$. Then*

$$p = (\bigcup_{n \in \mathbb{N}} A_n, \bigcup_{n \in \mathbb{N}} \mathcal{F}_n, \bigcup_{n \in \mathbb{N}} \mathcal{I}_n).$$

is a condition of \mathbb{P} satisfying $p \leq p_n$ for each $n \in \mathbb{N}$.

Proof. First note that $p \in \mathbb{P}$. (1), (2), (3) and (4) are clear. To get (5) Note that if $f_1, \dots, f_k \in \mathcal{F}_p$, $A \in \mathcal{I}_p$ and $\xi \in A_p$, then $f_1, \dots, f_k \in \mathcal{F}_n$, $A \in \mathcal{I}_n$ and $\xi \in A_n$ for some $n \in \mathbb{N}$. Then by (5) for p_n there is $\sigma \in \Sigma_{\xi, A}(A_n)$, with $f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma) \in \mathcal{F}_n$. So we may use the fact that $\Sigma_{\xi, B}(C) \subseteq \Sigma_{\xi, B}(C')$ whenever $C \subseteq C'$. To get (6) apply 2.7. The fact that $p \leq p_n$ for each $n \in \mathbb{N}$ is clear. \square

4. THE PARTIAL ORDER OF APPROXIMATIONS

Definition 4.1. *We define a partial order \mathbb{Q} which consists of conditions of the form $p = (A_p, \mathcal{F}_p, \mathcal{I}_p, \alpha_p, \mathcal{X}_p, \mathcal{P}_p)$ such that*

- (1) $(A_p, \mathcal{F}_p, \mathcal{I}_p) \in \mathbb{P}$
- (2) α_p is a countable ordinal,
- (3) $\mathcal{X}_p = \{x_\beta^p : \beta < \alpha_p\}$ is a countable dense subset of $\nabla \mathcal{F}_p$
- (4) \mathcal{P}_p is a countable family of pairs (L, R) of disjoint subsets α_p such that $\{x_\xi^p : \xi \in L\}$ and $\{x_\xi^p : \xi \in R\}$ are relatively discrete in $\nabla \mathcal{F}_p$ and

$$\overline{\{x_\xi^p : \xi \in L\}} \cap \overline{\{x_\xi^p : \xi \in R\}} \neq \emptyset.$$

We let $p \leq q$ if and only if $A_p \supseteq A_q$, $\mathcal{F}_p \supseteq \mathcal{F}_q$, $\mathcal{I}_p \supseteq \mathcal{I}_q$, $\alpha_p \geq \alpha_q$, $\mathcal{P}_p \supseteq \mathcal{P}_q$ and $x_\beta^p \restriction \nabla \mathcal{F}_q = x_\beta^q$ for any $\beta < \alpha_q$.

The point here is that we will make our spaces $\nabla \mathcal{F}_p$ quite complicated and rich in suprema of bounded sequences from $C(\nabla \mathcal{F}_p)$, but we will need to keep promises (that is, preserve elements of \mathcal{P}_p) about not separating some pairs of countable sets of points. If it is done in a sufficiently random manner, all the operators which are not weak multipliers are eliminated like in [10] or [9].

Definition 4.2. *Let $p, q \in \mathbb{Q}$. We say that they are isomorphic if there is a bijection $\tau : \omega_2 \rightarrow \omega_2$ which witnesses that $(A_p, \mathcal{F}_p, \mathcal{I}_p)$ and $(A_q, \mathcal{F}_q, \mathcal{I}_q)$ are isomorphic as elements of \mathbb{P} and*

- (1) $\alpha_p = \alpha_q$,
- (2) $x_\xi^p(f) = x_\xi^q(f \circ \phi(\tau))$ for all $f \in \mathcal{F}_p$ and all $\xi < \alpha_p = \alpha_q$,
- (3) If $f \in \mathcal{F}_p \cap \mathcal{F}_q$, then $x_\xi^p(f) = x_\xi^q(f)$ for all $\xi < \alpha_p$
- (4) $\mathcal{P}_q = \mathcal{P}_p$.

Lemma 4.3. *Suppose that $p, q \in \mathbb{Q}$ are isomorphic. Then there is $r = (A_p \cup A_q, \mathcal{F}_p \cup \mathcal{F}_q, \langle \mathcal{I}_p \cup \mathcal{I}_q \rangle, \alpha_p + \omega, \mathcal{X}, \mathcal{P}_p)$ which is a condition of \mathbb{Q} stronger than both p and q where $\mathcal{X} = \{x_\beta^r : \beta < \alpha_r\}$ consists of points which for every $f \in \mathcal{F}_q$ satisfy:*

$$x_\xi^r(f) = x_\xi^r(f \circ \phi(\tau)) \text{ for all } f \in \mathcal{F}_p, \xi < \alpha_p = \alpha_q.$$

Proof. By 3.4 $(A_p \cup A_q, \mathcal{F}_p \cup \mathcal{F}_q, \langle \mathcal{I}_p \cup \mathcal{I}_q \rangle) \leq (A_p, \mathcal{F}_p, \mathcal{I}_p), (A_q, \mathcal{F}_q, \mathcal{I}_q)$. Given $\beta < \alpha_p$ find $u \in K$ such that $(\Pi \mathcal{F}_p)(u) = x_\beta^p$. Let $v = \phi(\tau)^{-1}(u)$, then for all $f \in \mathcal{F}_p$ we have $f \circ \phi(\tau)(v) = f(u) = x_\beta^p(f) = x_\beta^q(f \circ \phi(\tau))$, and so $(\Pi \mathcal{F}_q)(v) = x_\beta^q$ by 2) of 4.2 and 2) of 3.3. Note that $v \cap Co(A_p \cap A_q) = u \cap Co(A_p \cap A_q)$ because τ is the identity on $A_p \cap A_q$. So by 2.4 there is an ultrafilter $w \in K$ which includes $u \cap Co(A_p)$ and $v \cap Co(A_q)$, hence $x_r^\beta = (\Pi \mathcal{F}_r)(w)$ belongs to $\nabla \mathcal{F}_r$ and satisfies the requirement of the lemma. So we are allowed for $\beta < \alpha$ to define $x_\beta^r(f) = x_\beta^p(f)$ if $f \in \mathcal{F}_p$ and $x_\beta^r(f) = x_\beta^q(f)$ if $f \in \mathcal{F}_q$.

Define $x_{\alpha+n}^r$ s to be points of some dense subset of $\nabla \mathcal{F}_r$ so that (3) of 4.1 will be satisfied. So to finish the proof we need to show (4) of 4.1, i.e., that the promises are preserved, i.e., that for each $(L, R) \in \mathcal{P}_r$ $\{x_\xi^r : \xi \in L\}$ and $\{x_\xi^r : \xi \in R\}$ are relatively discrete in $\nabla \mathcal{F}_r$ and

$$\overline{\{x_\xi^r : \xi \in L\}} \cap \overline{\{x_\xi^r : \xi \in R\}} \neq \emptyset.$$

in $\nabla \mathcal{F}_r$. Of course the canonical projections from $\nabla \mathcal{F}_r$ onto $\nabla \mathcal{F}_p$ or $\nabla \mathcal{F}_q$ are continuous and send x_β^r to x_β^p or x_β^q respectively. As the images are discrete, the preimages must be as well so for each $(L, R) \in \mathcal{P}_r$ $\{x_\xi^r : \xi \in L\}$ and $\{x_\xi^r : \xi \in R\}$ are relatively discrete in $\nabla \mathcal{F}_r$. Now assume that $(L, R) \in \mathcal{P}_r = \mathcal{P}_p = \mathcal{P}_q$. It is enough to prove that for each finite $\mathcal{F} \subseteq \mathcal{F}_r$, for every $\varepsilon > 0$ there are $\xi \in L$ and $\xi' \in R$ such that $|x_\xi(f) - x_{\xi'}(f)| < \varepsilon$ for all $f \in \mathcal{F}$. Fix \mathcal{F} and ε as above. Let $\tau : \omega_2 \rightarrow \omega_2$ be a bijection like in 4.2 witnessing the isomorphism of p and q . Let

$$\mathcal{G} = (\mathcal{F} \cap \mathcal{F}_p) \cup \{f \circ \phi(\tau^{-1}) : f \in \mathcal{F} \cap \mathcal{F}_q\} \subseteq \mathcal{F}_p.$$

By (4) of 4.1 find $\xi \in L$ and $\xi' \in R$ such that $|x_\xi^r(f) - x_{\xi'}^r(f)| < \varepsilon$ holds for each $f \in \mathcal{G}$. In particular for each $f \in \mathcal{F} \cap \mathcal{F}_p$ we have

$$|x_\xi^r(f) - x_{\xi'}^r(f)| = |x_\xi^p(f) - x_{\xi'}^p(f)| < \varepsilon.$$

But by (2) of 4.2, for $f \in \mathcal{F} \cap \mathcal{F}_q$ we have that

$$\begin{aligned} |x_\xi^r(f) - x_{\xi'}^r(f)| &= |x_\xi^q(f \circ \phi(\tau^{-1}) \circ \phi(\tau)) - x_{\xi'}^q(f \circ \phi(\tau^{-1}) \circ \phi(\tau))| = \\ &= |x_\xi^p(f \circ \phi(\tau^{-1})) - x_{\xi'}^p(f \circ \phi(\tau^{-1}))| = |x_\xi^p(g) - x_{\xi'}^p(g)| < \varepsilon, \end{aligned}$$

where g is some element of \mathcal{G} , which proves that (4) of 4.1 holds for r and completes the proof of the lemma. \square

Lemma 4.4. *Assume CH. Let $A \subseteq \omega_2$ be a countable set. There are ω_1 conditions p of \mathbb{P} with $A_p = A$.*

Proof. By 2.12 there are ω_1 functions in $C_I(K)$ which depend on A , so using the fact that there are continuum countable subsets of ω_1 we see that there are ω_1 possibilities for the family \mathcal{F}_p . There are also continuum many possibilities for the countable set in $\wp(A)$ which generates \mathcal{I}_p . Similar argument shows that there are continuum many possibilities for P_p and of course for α_p . Note that given a countable $\mathcal{F}_p \subseteq C_I(K)$, $\nabla \mathcal{F}_p$ is completely determined as a subset of $[0, 1]^{\mathcal{F}_p}$ which has cardinality continuum, so we have again continuum many possibilities for its countable dense subsets \mathcal{X}_p . This completes the proof. \square

Lemma 4.5. *Assume CH. \mathbb{Q} satisfies the ω_2 -c.c.*

Proof. Let $(p_\xi : \xi < \omega_2)$ be a family of elements of \mathbb{Q} . Using the Δ -system lemma for the family $(A_{p_\xi} : \xi < \omega_2)$, which holds under CH, we may assume that these sets form a Δ -system with root $\Delta < A_{p_\xi} \setminus \Delta$ of the same order type $\theta < \omega_1$ for each $\xi < \omega_2$. Again applying CH we may assume that $A_{p_\xi} \setminus \Delta < A_{p_\eta}$ for every $\xi < \eta < \omega_2$.

Consider the unique order-preserving bijections $\tau_\xi : A_{p_\xi} \rightarrow \theta$. And conditions of \mathbb{Q} of the form $q_\xi = (\theta, \mathcal{F}_\xi, \mathcal{I}_\xi, \alpha_\xi, \mathcal{X}_\xi, \mathcal{P}_\xi)$ where

- $\mathcal{F}_\xi = \{f \circ \phi(\tau_\xi) : f \in \mathcal{F}_{p_\xi}\},$
- $\mathcal{I}_\xi = \{\tau_\xi[A] : A \in \mathcal{I}_{p_\xi}\}.$
- $\alpha_\xi = \alpha_{p_\xi},$
- $\mathcal{X}_\xi = \{x_\beta^\xi : \beta < \alpha_\xi\}$ is a subset of $\nabla \mathcal{F}_\xi$ satisfying $x_\beta^\xi(f \circ \phi(\tau_\xi)) = x_\beta^{p_\xi}(f)$ for all $f \in \mathcal{F}_p.$
- $\mathcal{P}_\xi = \mathcal{P}_{p_\xi}.$

Using CH by 4.4 there are at most ω_1 conditions of \mathbb{Q} with the first coordinate equal to η . Thus we have $q_\xi = q_{\xi'}$ for distinct $\xi, \xi' < \omega_2$. Now an extension τ of $\tau_{\xi'}^{-1} \circ \tau_\xi$ witnesses the isomorphism between p_ξ and $p_{\xi'}$. \square

Lemma 4.6. *Suppose that $(p_n)_{n \in \mathbb{N}}$ is a sequence of conditions of \mathbb{Q} satisfying $p_{n+1} \leq p_n$ and $p_n = (A_n, \mathcal{F}_n, \mathcal{I}_n, \alpha_n, \mathcal{X}_n, \mathcal{I}_n)$ for each $n \in \mathbb{N}$. Then there is*

$$p = (\bigcup_{n \in \mathbb{N}} A_n, \bigcup_{n \in \mathbb{N}} \mathcal{F}_n, \bigcup_{n \in \mathbb{N}} \mathcal{I}_n, \alpha, \mathcal{X}, \bigcup_{n \in \mathbb{N}} \mathcal{P}_n)$$

which is a condition of \mathbb{P} satisfying $p \leq p_n$ for each $n \in \mathbb{N}$ where $\alpha = \sup_{n \in \mathbb{N}} \alpha_n$ and $\mathcal{X} = \{x_\beta : \beta < \alpha\}$ is a subset of $\nabla \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$ satisfying $x_\beta(f) = x_\beta^{p_n}(f)$ for $f \in \mathcal{F}_n$. In particular \mathbb{Q} is σ -closed.

Proof. First we need to prove the existence of x_β^p as in the lemma. Let $u_n \in K$ be such that $(\Pi \mathcal{F}_n)(u_n) = x_\beta^{p_n}$. Let $u \in K$ be a complete accumulation point of u_n 's in K . We claim that $x_\beta^p = (\Pi \mathcal{F}_n)(u)$ works. For $f \in \mathcal{F}_n$, for $k \geq n$ we have $f(u_k) = x_\beta^n$ by the assumption that $p_k \leq p_n$, so $f(u) = x_\beta^n$ must hold as well.

Now we claim that $p \in \mathbb{P}$ and $p \leq p_n$ for each $n \in \mathbb{N}$. It is enough to prove the latter.

(1), (2), (3) and (4) are clear. To get (5) Note that if $f_1, \dots, f_k \in \mathcal{F}_p$, $A \in \mathcal{I}_p$ and $\xi \in A_p$, then $f_1, \dots, f_k \in \mathcal{F}_n$, $A \in \mathcal{I}_n$ and $\xi \in A_n$ for some $n \in \mathbb{N}$. Then by (5) for p_n there is $\sigma \in \Sigma_{\xi, A}(A_n)$, with $f_1 \circ \phi(\sigma), \dots, f_k \circ \phi(\sigma) \in \mathcal{F}_n$. So we may use the fact that $\Sigma_{\xi, B}(C) \subseteq \Sigma_{\xi, B}(C')$ whenever $C \subseteq C'$. To get (6) apply 2.7. \square

5. ADDING SUPREMA OF DISJOINT SEQUENCES OF FUNCTIONS

Suppose that L is a compact space. We say that functions $f_n : L \rightarrow \mathbb{R}$ for $n \in \mathbb{N}$ are pairwise disjoint if $f_n(x)f_{n'}(x) = 0$ for all distinct $n, n' \in \mathbb{N}$ and all $x \in L$. $GR(f)$ will denote the graph of a function f . We need a simple lemma about pointwise sums of pairwise disjoint sequences of functions:

Lemma 5.1. *Suppose $f_n : L \rightarrow [0, 1]$ for $n \in \mathbb{N}$ are pairwise disjoint. Then:*

- (1) *If $(x, t) \in GR(\sum_{n \in \mathbb{N}} f_n)$, then there is $n' \in \mathbb{N}$ such that for all $m > n'$ we have $(x, t) \in GR(\sum_{n=0}^m f_n)$*

Lemma 5.2. *Suppose L is a metrizable, compact and connected space and that for each $k \leq m \in \mathbb{N}$ the sequences of functions $(f_n^k)_{n \in \mathbb{N}}$ in $C_I(L)$ are pairwise disjoint. Let $F : L \rightarrow [0, 1]^m$ be defined by*

$$F(x) = (\sum_{n \in \mathbb{N}} f_n^1(x), \dots, \sum_{n \in \mathbb{N}} f_n^m(x)) = \prod_{i \leq m} \sum_{n \in \mathbb{N}} f_n^i(x).$$

for every $x \in L$. Then the closure of the graph of F is a connected subspace of $L \times [0, 1]^m$.

Proof. Let X be the closure of the graph $GR(F)$ of F . For $l_1, \dots, l_m \in \mathbb{N}$ define $F^{l_1, \dots, l_m} : L \rightarrow [0, 1]^m$ by

$$F^{l_1, \dots, l_m}(x) = (\sum_{n=0}^{l_1} f_n^1(x), \dots, \sum_{n=0}^{l_m} f_n^m(x)),$$

for every $x \in L$. Now consider the set

$$Y = \bigcap_{(l_1, \dots, l_m) \in \mathbb{I}^m} \overline{\bigcup_{k \in \mathbb{N}} GR(F^{l_1(k), \dots, l_m(k)})},$$

where \mathbb{I} stands for the set of all strictly increasing sequences of positive integers.

Note that every sum in the above set is a connected set because the functions are continuous and there is a point where all of them are zero by compactness of L and the disjointness of the functions. It follows that the closures of the sums are connected and so that Y is connected as well. Thus it is enough to prove that $X = Y$.

First prove that $X \subseteq Y$. Let $y = (x, t_1, \dots, t_m) \in X = \overline{GR(F)}$. We will show that $y \in Y$. Let $(l_1, \dots, l_m) \in \mathbb{I}^m$. As L is metrizable, we can find a sequence $(y_n)_{n \in \mathbb{N}}$ converging to y such that $y_n \in GR(F)$. By 5.1 for each $n \in \mathbb{N}$ there is $n' \in \mathbb{N}$ such that $y_n \in GR(F^{l_1, \dots, l_m})$ for all $l > n'$ and hence there is $k \in \mathbb{N}$ such that $y_n \in GR(F^{l_1(k), \dots, l_m(k)})$. Hence all y_n s are in all the sums appearing in the definition of Y hence y is in all of their closures and so is in Y .

Now we show that $Y \subseteq X$. Suppose $y = (x, t_1, \dots, t_m) \notin X = \overline{GR(F)}$ and let $U \times V_1 \times \dots \times V_m$ be an open neighbourhood of y disjoint from $GR(F)$. By considering a slightly smaller set we may assume that $\overline{U} \times \overline{V}_1 \times \dots \times \overline{V}_m$ is disjoint from $GR(F)$. Moreover assume that V_i is separated from 0 if $t_i \neq 0$ for $1 \leq i \leq m$.

Let $k \in \mathbb{N}$. We will find $k'(k) > k$ such that for some choice of $l_1(k), \dots, l_m(k) \in \{k+1, k'(k)\}$ the graph $GR(F^{l_1(k), \dots, l_m(k)})$ is disjoint from $U \times V_1 \times \dots \times V_m$.

As k is arbitrary, we get that $\bigcup_{k \in \mathbb{N}} GR(F^{l_1(k), \dots, l_m(k)})$ is disjoint from $U \times V_1 \times \dots \times V_m$ proving that y is not in Y .

Note that if all t_1, \dots, t_m were bigger than 0, hence all V_1, \dots, V_m separated from 0, and $GR(F^{k+1, \dots, k+1})$ intersected $U \times V_1 \times \dots \times V_m$, then the graph $GR(F)$ intersected it as well, because $0 \leq F^{k+1, \dots, k+1}(x) = F(x)$ if all the coordinates of $F^{k+1, \dots, k+1}(x)$ are bigger than 0 by the disjointness of the functions in the original sequences.

On the other hand if all $t_1, \dots, t_m = 0$, then $F(x) = (s_1, \dots, s_m) \neq (0, \dots, 0)$, so there is $1 \leq i \leq m$ with $s_i \notin V_i$. Hence there is $k' > k$ such that $\sum_{n=1}^{k'} f_n^i(x) = s_i$ and so $GR(F^{k', \dots, k'})$ is disjoint from $U \times V_1 \times \dots \times V_{j_0} \times V_{j_0+1} \times \dots \times V_m$.

So, some of the values of t_1, \dots, t_m are 0 and some are not. We will continue the proof under the assumption that there is $1 \leq m_0 \leq m$ be such that $t_1 = \dots = t_{m_0} = 0$ and $t_{m_0+1}, \dots, t_m > 0$. This of course can easily be transformed into the general case with a different configuration.

Recall that we have that $0 \notin \overline{V}_{m_0+1}, \dots, \overline{V}_m$. Let $F_{m_0}^{k+1} : L \times [0, 1]^{m-m_0}$ be defined by

$$F_{m_0}^{k+1}(x) = (\sum_{n=1}^{k+1} f_n^{m_0+1}(x), \dots, \sum_{n=1}^{k+1} f_n^m(x))$$

Consider

$$E = (F_{m_0}^{k+1})^{-1}[\overline{V}_{m_0+1} \times \dots \times \overline{V}_m] \cap U$$

If E is empty then $(F^{k+1, \dots, k+1})^{-1}[\overline{V}_1, \dots, \overline{V}_m] \cap U$ is empty what is required. If E is nonempty, consider

$$\mathcal{U} = \{(f_n^1)^{-1}[[0, 1] \setminus \overline{V}_1], \dots, (f_n^{m_0})^{-1}[[0, 1] \setminus \overline{V}_{m_0}] : n \in \mathbb{N}\}.$$

$\bigcup \mathcal{U}$ must include E because otherwise there would be $x \in U$ with $F(x) \in V_1 \times \dots \times V_{m_0} \times \overline{V}_{m_0+1} \times \dots \times \overline{V}_m$ which would contradict the choice of $U \times V_1 \times \dots \times V_m$. But E is compact, so there is a finite $\mathcal{U}' \subseteq \mathcal{U}$ which covers E . Let $k' > k$ be such that $E \subseteq \bigcup \{(f_n^1)^{-1}[[0, 1] \setminus \overline{V}_1], \dots, (f_n^{m_0})^{-1}[[0, 1] \setminus \overline{V}_{m_0}] : 1 \leq n \leq k'\}$. This means that the graph $F^{k', \dots, k', k+1, \dots, k+1}$ is disjoint from $U \times V_1 \times \dots \times V_{m_0} \times \overline{V}_{m_0+1} \times \dots \times \overline{V}_m$ as required. \square

Definition 5.3. Suppose that L is a compact space and $f_n : L \rightarrow [0, 1]$ are pairwise disjoint continuous functions for $n \in A \subseteq \mathbb{N}$. Then

$$D((f_n)_{n \in A}) = \bigcup \{U : U \text{ is open and } \{n \in A : \text{supp}(f_n) \cap U \neq \emptyset\} \text{ is finite}\}$$

Suppose that $f_n^i : L \rightarrow [0, 1]$ for each $n \in A \subseteq \mathbb{N}$ are pairwise disjoint continuous functions for each $i \in B \subseteq \mathbb{N}$. Define $D((f_n^i)_{n \in A, i \in B}) = \bigcap_{i \in B} D((f_n)_{n \in A})$.

Lemma 5.4. Let L be a compact space. Suppose that for each $i \in \mathbb{N}$ the sequence $(f_n^i)_{n \in \mathbb{N}}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$.

- (1) $D((f_n^i)_{i, n \in \mathbb{N}})$ is dense in L
- (2) For each $i \in \mathbb{N}$ the function $\sum_{n \in \mathbb{N}} f_n^i$ is continuous on $D((f_n^i)_{n \in \mathbb{N}})$.

Proof. As shown in [10] $D((f_n)_{n \in \mathbb{N}})$ is always a dense and open subset of K . So $D((f_n^i)_{i, n \in \mathbb{N}})$ is dense by the Baire category theorem for compact Hausdorff spaces. Note that on $D((f_n^i)_{n \in \mathbb{N}})$ the infinite sum $\sum_{n \in \mathbb{N}} f_n^i$ is equal to some finite subsum, and so, is continuous. \square

Definition 5.5. [4.2.[10]] Suppose that L is a compact space, $M \subseteq L \times [0, 1]^\mathbb{N}$ and for each $i \in B \subseteq \mathbb{N}$ the sequence $(f_n^i)_{n \in A}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$ for some $A, B \subseteq \mathbb{N}$. We say that M is an extension of L by $(f_n^i)_{n \in \mathbb{N}, i \in B}$ if and only if M is the closure of the graph of the restriction

$$\prod \left\{ \sum_{n \in A} f_n^i : i \in B \right\} | D((f_n^i)_{n \in A, i \in B}).$$

Moreover we say that M as above is a strong extension of L by $(f_n^i)_{n \in A, i \in B}$ if and only if the graph of $\prod \{ \sum_{n \in A} f_n^i : i \in B \}$ is a subset of M .

Lemma 5.6. Suppose that L is a compact and connected space and M is a strong extension of L , then M is compact and connected as well.

Proof. It follows from 5.2 and 5.5 since the graph of $\prod \{ \sum_{n \in A} f_n^i : i \in B \}$ must be dense in M if it is a subset of M . \square

Lemma 5.7. *Suppose that L is compact and metrizable. Suppose that for each $i \in \mathbb{N}$ the sequence $(f_n^i)_{n \in \mathbb{N}}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$. Suppose that for each finite $a \subseteq \mathbb{N}$ the extension of L by $(f_n^i)_{n \in \mathbb{N}, i \in a}$ is strong. Then, the extension of L by $(f_n^i)_{n, i \in \mathbb{N}}$ is strong as well.*

Proof. Let $D = D((f_n^i)_{n, i \in \mathbb{N}})$. Suppose that the extension of L by $(f_n^i)_{n, i \in \mathbb{N}}$ is not strong and take a point

$$y = (x, t_1, \dots) \in GR(\prod_{n \in \mathbb{N}} \{ \sum f_n^i : i \in \mathbb{N} \})$$

which is not in the closure of

$$GR(F|D) = GR(\prod_{n \in \mathbb{N}} \{ \sum f_n^i \upharpoonright D((f_n^i)_{n, i \in \mathbb{N}}) : i \in \mathbb{N} \}).$$

Let U be an open neighbourhood of y disjoint from $GR(F|D)$. We may assume that U is open basic, so there is a finite $a \subseteq \mathbb{N}$ which determines U , so U must be disjoint from the projection of $GR(F|D)$ on these coordinates, in other words, assuming (which can be done without loss of generality) that $a = \{1, \dots, k\}$ for some $k \in \mathbb{N}$ we get

$$(x, t_1, \dots, t_k) \in GR_k(F) = GR(\prod_{n \in \mathbb{N}} \{ \sum f_n^j : j \leq k \})$$

and its open neighbourhood $U \subseteq L \times [0, 1]^k$ such that U is disjoint from

$$GR_k(F|D) = GR(\prod_{n \in \mathbb{N}} \{ \sum f_n^j \upharpoonright D((f_n^i)_{n, i \in \mathbb{N}}) : j \leq k \}).$$

So, to obtain a contradiction with the hypothesis, it is enough to prove that

$$GR_k(f|D_k) = GR(\prod_{n \in \mathbb{N}} \{ \sum f_n^i \upharpoonright D((f_n^i)_{n \in \mathbb{N}, i \leq k}) : i \leq k \}) \subseteq \overline{GR_k(f|D)},$$

since the left-hand side is dense in $GR_k(F)$ by the assumption of the lemma that finite-dimensional extensions are strong. So take any $(x', t'_1, \dots, t'_k) \in GR_k(f|D_k)$ and a neighbourhood $U' \subseteq D_k = D((f_n^i)_{n \in \mathbb{N}, i \leq k})$ of x' and neighbourhoods V_i of t'_i for $1 \leq i \leq k$. All functions $\sum_{n \in \mathbb{N}} f_n^i$ for $1 \leq i \leq k$ are continuous in D_k by (2) of 5.4, so we can find $U'' \subseteq U'$ such that

$$x' \in U'' \subseteq \bigcap_{1 \leq i \leq k} (\sum_{n \in \mathbb{N}} f_n^i)^{-1}[V_i].$$

Now, by (1) of 5.4, take any $y \in D \cap U''$, we have $(y, s_1, \dots, s_k) \in GR_k(F|D)$ for some (s_1, \dots, s_k) and $(y, s_1, \dots, s_k) \in U \times V_1 \times \dots \times V_k$, so there is no neighbourhood which can separate (x', t'_1, \dots, t'_k) from $GR_k(f|D)$ and so $GR_k(f|D_k) \subseteq \overline{GR_k(f|D)}$ as required. \square

Lemma 5.8. *Suppose that L is compact and metrizable. Suppose that for each $i \leq m$ the sequence $(f_n^i)_{n \in \mathbb{N}}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$. Then there is an infinite $A \subseteq \mathbb{N}$ such that for every infinite A' almost included in A the extension of L by $(f_n^i)_{n \in A', i \leq m}$ is a strong extension.*

Proof. Let d be a metric on L compatible with its topology. We construct $A = \{n_k : k \in \mathbb{N}\}$ so that $n_k < n_{k+1}$ for all $k \in \mathbb{N}$ and whenever $A' \subseteq \mathbb{N}$ is disjoint with

the intervals (n_k, n_{k+1}) for almost all $k \in \mathbb{N}$, then for every $\varepsilon > 0$, for every $x \in L$ there is

$$y \in D = \bigcap_{i \leq m} D((f_n^i)_{n \in A', i \leq m})$$

such that $d(x, y) < \varepsilon$ and $|\sum_{n \in \mathbb{N}} f_n^i(x) - \sum_{n \in \mathbb{N}} f_n^i(y)| < \varepsilon$ for each $i \leq m$.

We construct n_k s by induction. Suppose we have constructed $n_1 < \dots < n_k$. Using the compactness and the continuity of the functions $\sum_{n \in a} f_n^i$ for $i \leq m$ and $a \subseteq [0, n_k]$ find a finite open cover \mathcal{U}_k of L consisting of sets U such that

$$(1) \quad \text{diam}(\sum_{n \in a} f_n^i[U]) < 1/k, \quad \text{diam}(U) < 1/k,$$

for every $i \leq m$ and every $a \subseteq [0, n_k]$. Pick $y_U \in U \cap D$ for every $U \in \mathcal{U}_k$ and define n_{k+1} to be such a natural number bigger than n_k that

$$(2) \quad \forall U \in \mathcal{U}_k \quad \forall i \leq m \quad \forall n \geq n_{k+1} \quad f_n^i(y_U) = 0.$$

It can be found because of the disjointness of the functions. This completes the inductive construction.

Now take any $A' \subseteq \mathbb{N}$ disjoint with the intervals (n_k, n_{k+1}) for almost all $k \in \mathbb{N}$, fix $\varepsilon > 0$ and $x \in L$. Let k be big enough so that (n_k, n_{k+1}) is disjoint from A' , $1/k < \varepsilon$ and if $\sum_{n \in A'} f_n^i(x) > 0$, then $f_n^i(x) > 0$ for some $n \leq n_k$ and $n \in A'$. In particular

$$\forall i \leq m \quad \sum_{n \in A'} f_n^i(x) = \sum_{n \in a} f_n^i(x),$$

where $a = A' \cap [0, n_k]$. Find $U \in \mathcal{U}_k$ such that $x \in U$ and y_U from the construction. In particular $d(y_U, x) < \varepsilon$. For all $i \leq m$ we have

$$\sum_{n \in A'} f_n^i(y_U) = \sum_{n \in a} f_n^i(y_U) + \sum_{n \in A' \cap (n_k, n_{k+1})} f_n^i(y_U) + \sum_{n \in A' \cap [n_{k+1}, \infty)} f_n^i(y_U).$$

But the two last terms are zero by the choice of A' and by the choice (2) of n_{k+1} in the inductive construction, so for all $i \leq m$ we have $\sum_{n \in A'} f_n^i(y_U) = \sum_{n \in a} f_n^i(y_U)$ and hence $|\sum_{n \in A'} f_n^i(x) - \sum_{n \in A'} f_n^i(y_U)| < 1/k < \varepsilon$ for each $i \leq m$ by (1). \square

Lemma 5.9. *Suppose that L is compact and metrizable. Suppose that for each $i \in \mathbb{N}$ the sequence $(f_n^i)_{n \in \mathbb{N}}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$. Then there is an infinite $A \subseteq \mathbb{N}$ such that for every infinite A' almost included in A the extension of L by $(f_n^i)_{n \in A', i \in \mathbb{N}}$ is a strong extension.*

Proof. Enumerate all finite subsets of \mathbb{N} as $\{a_k : k \in \mathbb{N}\}$. By induction construct almost decreasing sequence $(A_k)_{k \in \mathbb{N}}$ of infinite subsets of \mathbb{N} such that A_{k+1} is almost included in A_k for each $k \in \mathbb{N}$ and A_k satisfies 5.8 for a_k instead of $\{1, \dots, m\}$, that is, for every A' almost included in A_k the extension of L by $(\sum_{n \in A'} f_n^i)_{n \in \mathbb{N}, i \in a_k}$ is strong. Now let A be the diagonalization of A_k s, that is, an infinite $A \subseteq \mathbb{N}$ which is almost included in A_k for each $k \in \mathbb{N}$. It is clear that if A' is almost included in A , then it is almost included in each A_k for $k \in \mathbb{N}$. Now apply lemma 5.7 to conclude that the extension of L by $(f_n^i)_{n \in A', i \in \mathbb{N}}$ is strong. \square

Lemma 5.10. *Suppose that L is compact and metrizable. Suppose that for each $i \in \mathbb{N}$ the sequence $(f_n^i)_{n \in \mathbb{N}}$ consists of pairwise disjoint continuous functions from L into $[0, 1]$. Let $x_n, y_n \in L$ for $n \in \mathbb{N}$ be such that*

$$\overline{\{x_n : n \in \mathbb{N}\}} \cap \overline{\{y_n : n \in \mathbb{N}\}} \neq \emptyset.$$

Then there is an infinite $A \subseteq \mathbb{N}$ such that for every infinite A' almost included in A we have

$$\overline{\{x'_n : n \in \mathbb{N}\}} \cap \overline{\{y'_n : n \in \mathbb{N}\}} \neq \emptyset,$$

where x'_n is the point of $GR(\Pi_{i \in \mathbb{N}} \sum_{n \in A'} f_n^i)$ whose L -coordinate is x_n and y'_n is the point of $GR(\Pi_{i \in \mathbb{N}} \sum_{n \in A'} f_n^i)$ whose L -coordinate is y_n .

Proof. Let d be a metric on L compatible with the topology. By going to a subsequence and possibly renumeraling the points we may assume that

$$1) \quad d(x_n, y_n) < 1/n.$$

We construct two strictly increasing sequences $A = \{n_k : k \in \mathbb{N}\}$ and $(l_k : k \in \mathbb{N})$ such that whenever $A' \cap (n_k, n_{k+1}) = \emptyset$ then, for all $i \leq k$ we have

$$2) \quad \left| \sum_{n \in A'} f_n^i(x_{l_k}) - \sum_{n \in A'} f_n^i(y_{l_k}) \right| < 1/k.$$

This will be enough, since if A' is almost included in A , the above will hold for almost all $k \in \mathbb{N}$; if $\{(x_{l_k}, \Pi_{i \in \mathbb{N}} \sum_{n \in A'} f_n^i(x_{l_k})) : k \in \mathbb{N}\}$ were separated from $\{(y_{l_k}, \Pi_{i \in \mathbb{N}} \sum_{n \in A'} f_n^i(y_{l_k})) : k \in \mathbb{N}\}$, it could be done by clopen basic sets, which involve only finitely many coordinates, so there would be k_0 such that $\{(x_{l_k}, \Pi_{i \leq k_0} \sum_{n \in A'} f_n^i(x_{l_k})) : k \in \mathbb{N}\}$ were separated from $\{(y_{l_k}, \Pi_{i \leq k_0} \sum_{n \in A'} f_n^i(y_{l_k})) : k \in \mathbb{N}\}$. But 2) implies that these points of $L \times [0, 1]^{k_0}$ are arbitrarily close.

So let us focus on proving 2). The construction of $\{n_k : k \in \mathbb{N}\}$ and $(l_k : k \in \mathbb{N})$ is by induction. Suppose that we are done up to k . $\sum_{n \in a} f_n^i$ are uniformly continuous for each $i \leq k$ and each $a \subseteq [0, n_k]$. So find $\delta > 0$ such that if $d(x, y) < \delta$ then $|\sum_{n \in a} f_n^i(x) - \sum_{n \in a} f_n^i(y)| < 1/k$ holds for each $i \leq k$ and each $a \subseteq [0, n_k]$. Take l_{k+1} such that $\delta > 1/l_{k+1}$, $l_{k+1} > l_k$ and take $n_{k+1} > n_k$ big enough so that for all $i \leq k$ we have

$$3) \quad \sum_{n \geq n_{k+1}} f_n^i(x_{l_{k+1}}) = \sum_{n \geq n_{k+1}} f_n^i(y_{l_{k+1}}) = 0.$$

This completes the inductive construction.

Now, suppose that $A' \subseteq \mathbb{N}$ is such that $A' \cap (n_k, n_{k+1}) = \emptyset$. Put $a = A' \cap [0, n_k]$. By 1) and 3) for every $i \leq k+1$ we have

$$\begin{aligned} & \left| \sum_{n \in A'} f_n^i(x_{l_{k+1}}) - \sum_{n \in A'} f_n^i(y_{l_{k+1}}) \right| = \\ & \left| \sum_{n \in a} f_n^i(x_{l_{k+1}}) - \sum_{n \in a} f_n^i(y_{l_{k+1}}) + \sum_{n \geq n_{k+1}} f_n^i(x_{l_{k+1}}) - \sum_{n \geq n_{k+1}} f_n^i(y_{l_{k+1}}) \right| = \\ & \left| \sum_{n \in a} f_n^i(x_{l_{k+1}}) - \sum_{n \in a} f_n^i(y_{l_{k+1}}) \right| < 1/k \end{aligned}$$

what was required. \square

Given $p \in \mathbb{P}$ we can add to \mathcal{F}_p suprema of continuous functions on $\nabla \mathcal{F}_p$ in such a way that they cannot be destroyed when we pass to bigger conditions $q \leq p$.

Definition 5.11. Suppose that $\mathcal{F} \subseteq C_I(K)$ and $f_n \in C_I(\nabla\mathcal{F})$ form a bounded sequence of functions, and let $h \in C(\nabla\mathcal{F})$ be the supremum of f_n s. Then we say that h is an indestructible supremum of f_n s if and only if for any $\mathcal{G} \subseteq C_I(K)$ such that $\mathcal{F} \subseteq \mathcal{G}$ we have that $h \circ \pi_{\mathcal{F}\mathcal{G}}$ is the supremum of $(f_n \circ \pi_{\mathcal{F}\mathcal{G}})_{n \in \mathbb{N}}$ in $C(\nabla\mathcal{G})$.

Lemma 5.12. Suppose that $\mathcal{F} \subseteq C_I(K)$ and $f_n \in C_I(\nabla\mathcal{F})$ form a bounded sequence of functions, Then there is a $g \in C_I(K)$ such that in $C(\nabla(\mathcal{F} \cup \{g\}))$ there is an indestructible supremum of $f_n \circ \pi_{\mathcal{F}, \mathcal{F} \cup \{g\}}$ s.

Proof. Consider $f_n \circ \Pi\mathcal{F} : K \rightarrow \mathbb{R}$. Since K is extremely disconnected, $C(K)$ is a complete lattice and so we have $g = \sup(f_n \circ \Pi\mathcal{F})$ in $C(K)$. Now, the supremum h of $f_n \circ \pi_{\mathcal{F}, \mathcal{F} \cup \{g\}}$ s is just taking g 's coordinate in $\nabla(\mathcal{F} \cup \{g\})$. First we will note that $h \geq f_n \circ \pi_{\mathcal{F}, \mathcal{F} \cup \{g\}}$ for all $n \in \mathbb{N}$. Take $t \in \nabla(\mathcal{F} \cup \{g\})$ and $x \in K$ such that $t = \Pi(\mathcal{F} \cup \{g\})$, we have

$$(f_n \circ \pi_{\mathcal{F}, \mathcal{F} \cup \{g\}})(t) = f_n \circ \pi_{\mathcal{F}, \mathcal{F} \cup \{g\}} \circ \Pi(\mathcal{F} \cup \{g\})(x) = f_n \circ \Pi(\mathcal{F})(x) \geq g(x) = t(g) = h(t),$$

as needed.

Now, suppose $\mathcal{H} = \mathcal{F} \cup \{g\} \subseteq \mathcal{G} \subseteq C_I(K)$ and that $h \circ \pi_{\mathcal{H}\mathcal{G}}$ is not the supremum of $f_n \circ \pi_{\mathcal{F}\mathcal{G}}$ s in $C(\nabla\mathcal{G})$. Let $f \in C_I(\nabla\mathcal{G})$ be a function which witnesses it. That is $f_n \circ \pi_{\mathcal{F}, \mathcal{G}} \leq f$ but there is a nonempty open set $U \subseteq \nabla\mathcal{G}$ such that $f|U < h \circ \pi_{\mathcal{H}, \mathcal{G}}|U$. Note that then we have

$$f_n \circ \Pi\mathcal{F} = f_n \circ \pi_{\mathcal{F}, \mathcal{G}} \circ \Pi\mathcal{G} \leq f \circ \Pi\mathcal{G}.$$

and

$$f \circ \Pi\mathcal{G}|V < h \circ \pi_{\mathcal{H}, \mathcal{G}} \circ \Pi\mathcal{G}|V = g|V,$$

where $V = (\Pi\mathcal{G})^{-1}[U]$. But this contradicts the fact that g is the supremum of $f_n \circ \Pi\mathcal{F}$ s in $C(K)$. \square

Lemma 5.13. Suppose $\mathcal{F} \subseteq C_I(K)$ and that for each $i \in \mathbb{N}$ the sequences $(f_n^i)_{n \in \mathbb{N}}$ of elements of $C_I(\nabla\mathcal{F})$ are pairwise disjoint. Suppose that all elements of \mathcal{F} depend on a set $A \subseteq \omega_2$ and that $\{d_\xi : \xi \in A\} \subseteq \mathcal{F}$. Suppose that $f^i \in C_I(K)$ is the supremum of $(f_n^i \circ \Pi\mathcal{F})_{n \in \mathbb{N}}$. Then $\nabla(\mathcal{F} \cup \{f^i : i \in \mathbb{N}\})$ is an extension of $\nabla\mathcal{F}$ by $(f_n^i)_{n \in \mathbb{N}}$.

Proof. First we will prove that if $D \subseteq \nabla\mathcal{F}$ is dense and open in $\nabla\mathcal{F}$, then $(\Pi\mathcal{F})^{-1}[D]$ is dense and open in K . Suppose not, then there are $\xi_1, \dots, \xi_n \in \omega_2$ and $\varepsilon_1, \dots, \varepsilon_n \in \{-1, 1\}$ such that

$$[a_{\xi_1}^{\varepsilon_1}] \cap \dots \cap [a_{\xi_n}^{\varepsilon_n}] \cap (\Pi\mathcal{F})^{-1}[D] = \emptyset,$$

Where $a^1 = a$ and a^{-1} is the complement of a in the Boolean algebra $Fr(\omega_2)$. As elements of \mathcal{F} depend on A , the characteristic function of $(\Pi\mathcal{F})^{-1}[D]$ depend on A , and hence we may assume that $\xi_i \in A$ for all $i \leq n$.

$$E = \bigcap_{1 \leq i \leq n} \{t \in \nabla\mathcal{F} : |t(d_{\xi_i}) - \varepsilon_i| < 1/4\}$$

is a nonempty basic open set in $\nabla\mathcal{F}$, since $\Pi\{d_{\xi_i} : 1 \leq i \leq n\}$ s are onto $[0, 1]^{\{d_{\xi_i} : 1 \leq i \leq n\}}$. So $E \cap D \neq \emptyset$ by the density of D . But if $x \in (\Pi\mathcal{F})^{-1}[E \cap D]$ then $x \in [a_{\xi_i}^{\varepsilon_i}]$ for each $i \leq n$ a contradiction.

In particular we can conclude that for each $i \in \mathbb{N}$ we have that $(\Pi\mathcal{F})^{-1}[D((f_n^i)_{n \in \mathbb{N}})]$ is dense open in K and so

$$(\Pi\mathcal{F})^{-1}\left[\bigcap_{i \in \mathbb{N}} D((f_n^i)_{n \in \mathbb{N}})\right] = \bigcap_{i \in \mathbb{N}} (\Pi\mathcal{F})^{-1}[D((f_n^i)_{n \in \mathbb{N}})]$$

is dense in K and hence $\nabla(\mathcal{F} \cup \{f^i : i \in \mathbb{N}\})$ is the closure of

$$\Pi(F \cup \{f^i : i \in \mathbb{N}\})[(\Pi\mathcal{F})^{-1}[\bigcap_{i \in \mathbb{N}} D((f_n)_{n \in \mathbb{N}})]].$$

Note that for each $i \in \mathbb{N}$ we have $(\Pi\mathcal{F})^{-1}[D((f_n^i)_{n \in \mathbb{N}})] \subseteq D((f_n^i \circ \Pi\mathcal{F})_{n \in \mathbb{N}})$, because the preimages of disjoint sets are disjoint. By 5.4 (2), f^i coincides with $\sum_{n \in \mathbb{N}} f_n^i \circ \Pi\mathcal{F}$ on $D((f_n^i \circ \Pi\mathcal{F})_{n \in \mathbb{N}})$, so $\nabla(\mathcal{F} \cup \{f^i : i \in \mathbb{N}\})$ is the closure of

$$\Pi(\mathcal{F} \cup \{\sum_{n \in \mathbb{N}} f_n^i \circ \Pi\mathcal{F} : i \in \mathbb{N}\})[(\Pi\mathcal{F})^{-1}[\bigcap_{i \in \mathbb{N}} D((f_n^i)_{n \in \mathbb{N}})]]$$

which is exactly the closure of the graph of $\Pi_{i \in \mathbb{N}} \sum_{n \in \mathbb{N}} f_n^i$ restricted to $\bigcap_{i \in \mathbb{N}} D((f_n^i)_{n \in \mathbb{N}})$ which completes the proof of the lemma. \square

6. THE MAIN EXTENSION LEMMA

Lemma 6.1. *Suppose $f : K \rightarrow L$ is a continuous function and $(f_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint functions in $C_I(L)$. Then $(f_n \circ f)_{n \in \mathbb{N}}$ is pairwise disjoint as well.*

Lemma 6.2. *Suppose that $\phi : K \rightarrow K$ is a homeomorphism of K . Suppose that f is the supremum in $C(K)$ of a bounded sequence $(f_n)_{n \in \mathbb{N}}$, then $f \circ \phi$ is the supremum in $C(K)$ of $(f_n \circ \phi)_{n \in \mathbb{N}}$.*

Proof. A homeomorphism of K induces an order-preserving isometry of $C(K)$, so the supremum must be preserved. \square

Lemma 6.3. *Let p be in \mathbb{Q} . Suppose that*

- (1) $(f_n)_{n \in \mathbb{N}}$ is a sequence of pairwise disjoint functions in $C_I(\nabla\mathcal{F}_p)$,
- (2) $\{\xi_m : m \in \mathbb{N}\} \subseteq \alpha_p$ is such that $f_n(x_{\xi_m}^p) = 0$ for each $n, m \in \mathbb{N}$.

Then there is an infinite $A \subseteq \mathbb{N}$ such that for each infinite $A' \subseteq A$ there is $q_{A'} \leq p$ in \mathbb{Q} such that:

- (a) *there is an $f \in C_I(\nabla\mathcal{F}_{q_{A'}})$ which is the indestructible supremum of $(f_i \circ \pi_{\mathcal{F}_p, \mathcal{F}_{q_{A'}}})_{i \in A'}$,*
- (b) $(\{x_{\xi_n}^{q_{A'}} : n \in A'\}, \{x_{\xi_n}^{q_{A'}} : n \notin A'\}) \in \mathcal{P}_{q_{A'}}$.

Proof. First, we will find an auxiliary condition s using 4.6 for a sequence $(p_n)_{n \in \mathbb{N}}$ starting with $p_1 = p$. Let $\Theta = (\Theta_1, \Theta_2) : \mathbb{N} \rightarrow \mathbb{N} \times \mathbb{N}$ be a bijection such that $\Theta_2(n), \Theta_1(n) \leq n$ holds for each $n \in \mathbb{N}$. The sequence $(p_n)_{n \in \mathbb{N}}$ is constructed by induction together with a sequence of bijections $\Gamma_n : \mathbb{N} \rightarrow A_{p_n}$. Having constructed p_n we find

$$\eta_n = \Gamma_{\Theta_1(n)}(\Theta_2(n)) \in A_{p_{\Theta_1(n)}} \subseteq A_{p_n}.$$

Now consider an bijection $\tau_n : \omega_2 \rightarrow \omega_2$ such that $\tau_n \upharpoonright \eta_n = Id_{\eta_n}$ and $\tau_n[A_{p_n} \setminus \eta_n] > \sup(A_p)$. Construct a condition p'_n , transporting all the appropriate objects by functions induced by τ_n , so that τ_n witnesses that p_n and p'_n are isomorphic. Now let $p_{n+1} \leq p_n, p'_n$ be the amalgamation like in 4.3. In particular for $f \in \mathcal{F}_{p_n}$ for every $\beta < \alpha_{p_n}$ we have:

$$+) \quad x_{\beta}^{p_{n+1}}(f) = x_{\beta}^{p_{n+1}}(f \circ \phi(\tau_n)).$$

Finally let $\Gamma_{n+1} : \mathbb{N} \rightarrow A_{p_{n+1}}$ be any bijection between these sets. Let s be the lower bound of $(p_n)_{n \in \mathbb{N}}$ obtained using 4.6.

We will enrich several coordinates of s to obtain the conditions $q_{A'}$. We maintain $A_{q_{A'}} = A_s$. The main change will be adding the supremum of the f_n s which yields a bigger change in \mathcal{F}_s in order to prove (5) of 4.1, this also implies the necessity of changing \mathcal{I}_s , and \mathcal{X}_s and of course we will add the promise from 6.3 to \mathcal{P}_s . The coordinate $\alpha_{q_{A'}}$ will become $\alpha_s + \omega$.

Claim 0: Note that $\tau_n \in \Sigma_{\eta_n, A_{p_n}}(A_s)$.

Notation: \mathbb{I} will denote the set of all finite strictly increasing sequences of the form $n_1 < \dots < n_k$ where $n_1, \dots, n_k \in \mathbb{N}$ for $k \in \mathbb{N}$. The empty sequence \emptyset also belongs to \mathbb{I} .

Claim 1: Suppose $(n_1, \dots, n_k) \in \mathbb{I}$. If $x, y \in K$ are such that $\Pi\mathcal{F}_s(x) = \Pi\mathcal{F}_s(y)$, then $\Pi\mathcal{F}_p \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})(x) = \Pi\mathcal{F}_p \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})(y)$.

Proof of the claim: Let $f \in \mathcal{F}_p \subseteq \mathcal{F}_{p_{n_1}}$, so $f \circ \phi(\tau_{n_1}) \in \mathcal{F}_{p_{n_1}+1} \subseteq \mathcal{F}_{p_{n_2}}$. Continuing this argument, we get $f \circ \phi(\tau_{n_1}) \circ \phi(\tau_{n_2}) \in \mathcal{F}_{p_{n_3}}$, and so on, until $f \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k}) \in \mathcal{F}_s$. Now the claim follows directly from its hypothesis.

The above claim justifies the introduction of the following:

Notation: If $(n_1, \dots, n_k) \in \mathbb{I}$, then $f_n^{n_1, \dots, n_k} : \nabla F_s \rightarrow [0, 1]$ is a function satisfying:

$$f_n^{n_1, \dots, n_k} \circ \Pi\mathcal{F}_s = f_n \circ \Pi\mathcal{F}_p \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k}).$$

The continuity of the function above follows from the fact that continuous surjections between compact spaces are quotient maps (see [3]).

Claim 2: For every $(n_1, \dots, n_k) \in \mathbb{I}$ and every $n_{k+1} > n_k$, we have

$$f_n^{n_1, \dots, n_k, n_{k+1}} \circ (\Pi\mathcal{F}_s) = f_n^{n_1, \dots, n_k} \circ (\Pi\mathcal{F}_s) \circ \phi(\tau_{n_{k+1}}).$$

Claim 3: For every $(n_1, \dots, n_k) \in \mathbb{I}$ the sequence $(f_n^{n_1, \dots, n_k})_{n \in \mathbb{N}}$ is pairwise disjoint.

Proof of the claim: Use 6.1 and the fact that $\Pi\mathcal{F}_s$ is onto ∇F_s .

Claim 4: For every $(n_1, \dots, n_k) \in \mathbb{I}$ and every $m \in \mathbb{N}$ we have $f_n^{n_1, \dots, n_k}(x_{\xi_m}^s) = 0$.

Proof of the claim: We prove it by induction on $k \in \mathbb{N}$. For $k = 0$, the claim is our assumption that $f(x_{\xi_m}^p) = 0$ and 6.1. Suppose we have proved the claim up to k . Let $t_m \in K$ be such that $(\Pi\mathcal{F}_s)(t_m) = x_{\xi_m}^s$. We have

$$\begin{aligned} f_n^{n_1, \dots, n_k}(x_{\xi_m}^s) &= f_n^{n_1, \dots, n_k} \circ \Pi\mathcal{F}_s(t_m) = \\ &= f_n \circ \Pi\mathcal{F}_p \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})(t_m). \end{aligned}$$

If $f \in \mathcal{F}_p$, then $f(t_m) = x_{\xi_m}^s(f) = x_{\xi_m}^{p_{n_1}}(f) = x_{\xi_m}^{p_{n_1}+1}(f \circ \phi(\tau_{n_1})) = x_{\xi_m}^{p_{n_2}}(f \circ \phi(\tau_{n_1}))$ by +). Continuing this argument we get that

$$f(t_m) = x_{\xi_m}^s(f \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})) = f \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})(t_m).$$

So,

$$\begin{aligned} 0 &= f_n(x_{\xi_m}^p) = (f_n \circ \Pi\mathcal{F}_p)(t_m) = f_n \circ \Pi\mathcal{F}_p \circ \phi(\tau_{n_1}) \circ \dots \circ \phi(\tau_{n_k})(t_m) = \\ &= f_n^{n_1, \dots, n_k} \circ \Pi\mathcal{F}_s(t_m) = f_n^{n_1, \dots, n_k}(x_{\xi_m}^s). \end{aligned}$$

Notation: If $(n_1, \dots, n_k) \in \mathbb{I}$, and $A \subseteq \mathbb{N}$ then $f_A^{n_1, \dots, n_k}$ is the supremum of the pairwise disjoint sequence $(f_n^{n_1, \dots, n_k} \circ \Pi\mathcal{F}_s)_{n \in A}$ in $C_I(K)$.

Claim 5: If $(n_1, \dots, n_k) \in \mathbb{I}$, and $n_{k+1} > n_k$ and $A \subseteq \mathbb{N}$ then, $f_A^{n_1, \dots, n_k} \circ \phi(\tau_{n_{k+1}}) = f_A^{n_1, \dots, n_k, n_{k+1}}$.

Proof of the claim: This follows from 6.2 and claim 2.

Now, applying 5.9 and 5.10 find an infinite $A \subseteq \mathbb{N}$ such that whenever A' is almost included in A , then if M is an extension of ∇F_s by $(f_n^{n_1, \dots, n_k})_{n \in A', (n_1, \dots, n_k) \in \mathbb{I}}$, then M is a strong extension which preserves all the promises $(L, R) \in \mathcal{P}_q$ in the sense of 5.10. Moreover, we can assume, by choosing a convergent subsequence of $(x_{\xi_m}^s)_{m \in A}$ that $(x_{\xi_m}^s)_{m \in A}$ converges to a point which is also the limit of a convergent sequence $(x_{\xi_m}^s)_{m \in B}$ for some $B \subseteq \mathbb{N} \setminus A$.

Notation: If A' is almost included in an A as above, then we define

$$q_{A'} = (A_s, \mathcal{F}_s \cup \{f_{A'}^{n_1, \dots, n_k} : (n_1, \dots, n_k) \in \mathbb{I}\}, \mathcal{I}, \alpha_s + \omega, \mathcal{X}_{q_{A'}}, \mathcal{P}_s \cup \{L', R'\}),$$

where

- (1) $\mathcal{I} = \mathcal{I}_{q_{A'}}$ is the ideal of subsets of A_s generated by the sets A_{p_n} for $n \in \mathbb{N}$.
- (2) For each $\beta < \alpha_{q_{A'}} = \alpha_s$ the point $x_{\beta}^{q_{A'}}$ is the point of

$$GR(\Pi_{(n_1, \dots, n_k) \in \mathbb{I}} \sum_{n \in A'} f_n^{n_1, \dots, n_k})$$

above x_{β}^s .

- (3) For $\beta \in [\alpha_s, \alpha_s + \omega)$ we choose the points x_{β} so that they form a dense subset of $\nabla \mathcal{F}_{q_{A'}}$,
- (4) $L' = \{\xi_m : m \in A'\}$ and $R' = \{\xi_m : m \in B\}$.

We will prove that $q_{A'} \in \mathbb{Q}$. First we will check that $(A_s, \mathcal{F}_{q_{A'}}, \mathcal{I})$ is a condition of \mathbb{P} . Certainly (1)-(3) is true, and (4) follows from the fact that $\mathcal{F}_s = \bigcup_{n \in \mathbb{N}} \mathcal{F}_{p_n}$ by 3.5, the fact that f_n s depend on A_{p_1} and by 2.13. To check (5) we will prove the following:

Claim 6: Let $\xi \in A_s$, $A \in \mathcal{I}$ and let f_1, \dots, f_k for $k \in \mathbb{N}$ be functions in $\mathcal{F}_{q_A'}$ and $n \in \mathbb{N}$ such that $A \subseteq A_{p_n}$, $\xi \in A_{p_n}$ and $f_1, \dots, f_k \in \mathcal{F}_{p_n}$. Let $m \in \mathbb{N}$ be such that $\Gamma_n(m) = \xi$. Let $n' \geq n$ such that $\Theta(n') = (\Theta_1(n'), \Theta_2(n')) = (n, m)$. Then $\tau_{n'} \in \Sigma_{\xi, A}(A_s)$ and $f_i \circ \phi(\tau_{n'}) \in F_{q_A'}$ for each $1 \leq i \leq k$.

Proof of the claim: By the choice of Θ , there is $n' \geq n$ such that

$$\Theta(n') = (\Theta_1(n'), \Theta_2(n')) = (n, m),$$

so $\eta_{n'} = \xi$. Since $p_{n'+1}$ is an amalgamation of two isomorphic conditions $p_{n'}$ and $p_{n'}$, and the isomorphism is witnessed by $\tau_{n'}$ satisfying $\tau_{n'} \restriction \xi = Id_{\xi}$ and $\tau_{n'}[A_{p_{n'}} \setminus \xi] > \sup(A_{p_{n'}}) \geq \sup(A)$, since $\eta_{n'} = \xi$, we have $f_1 \circ \phi(\tau_{n'}), \dots, f_k \circ \phi(\tau_{n'}) \in \mathcal{F}_{n'+1} \subseteq \mathcal{F}_s \subseteq \mathcal{F}_{q_{A'}}$. This completes the proof of the claim.

Now (5) of 3.2 follows from the above claim and Claim 5. By the choice of A based on 5.9 the space $\nabla \mathcal{F}_{q_{A'}}$ is a strong extension of ∇F_s by $(f_n^{n_1, \dots, n_k})_{n \in A', (n_1, \dots, n_k) \in \mathbb{I}}$ and so is connected by Lemma 5.6 since $\nabla \mathcal{F}_s$ was connected. This gives 6) of the definition 3.2.

Now check Definition 4.1. We just checked (1), (2) is trivial, (3) follows from the fact that the extension is strong and from the definition of $\mathcal{X}_{q_{A'}}$. Now let us check (4) of 4.1. If $(L, R) \in \mathcal{P}_s$, then $\{x_{\beta}^{q_{A'}} : \beta \in L\} \cap \{x_{\beta}^{q_{A'}} : \beta \in R\} \neq \emptyset$ by the choice

of $A \subseteq \mathbb{N}$ based on 5.10. Also

$$++)) \quad \{x_{\xi_m}^{q_{A'}} : m \in A'\} \cap \{x_{\xi_m}^{q_{A'}} : m \in B\} \neq \emptyset$$

because $\prod_{(n_1, \dots, n_k) \in \mathbb{I}} \sum_{n \in A'} f_n^{n_1, \dots, n_k}(x_{\xi_m}^q) = 0$ for all $n, m \in \mathbb{N}$ by Claim 4 and so $x_{\xi_m}^{q_{A'}}$ is $x_{\xi_m}^q$ followed by the sequence consisting only of zeros, hence the choice of B gives us $++))$ and (b) of the lemma.

Clearly f^\emptyset is the indestructible supremum of $(f_n)_{n \in A'}$ which completes the proof of the lemma. \square

7. CONTINUOUS FUNCTIONS IN V AND $V[G]$

In this and in the following section we will employ the partial order \mathbb{Q} as a forcing notion (see [13], [7]). We start with a set-theoretic universe V which satisfies the CH and we will consider its generic extension $V[G]$ where G is a \mathbb{Q} -generic over V .

By 4.6 and 4.5, forcing with \mathbb{Q} does not collapse cardinals if CH holds in V and does not add any new countable subsets of the universe (see [Ku]), i.e., the ground set-theoretic universe V and the generic extension $V[G]$ have the same countable sets of V and the same cardinals.

This, in particular, means that the completion $Co(\omega_2)$ of the free Boolean algebra $Fr(\omega_2)$ with ω_2 -generators is the same in both of the universes V and $V[G]$. The Stone space K^* of $Co(\omega_2)$ is bigger in $V[G]$, but K is dense in it, so every $f \in C(K)$ as a uniformly continuous function, uniquely extends to a function $f^* \in C(K^*)$. If $\mathcal{F} \subseteq C_I(K)$, then \mathcal{F}^* will denote $\{f^* : f \in \mathcal{F}\}$.

As G is a \mathbb{Q} -generic filter, for every two $p, p' \in G$ there is $q \leq p, p'$ with $q \in G$ and $G \cap D \neq \emptyset$ for any dense set $D \subseteq \mathbb{Q}$ in the ground set-theoretic universe V . In the generic extension $V[G]$ we consider $\mathcal{F}_G = \bigcup \{\mathcal{F}_p : p \in G\}$ and the compact space $L = \nabla \mathcal{F}_G^*$ with the subset $\mathcal{X} = \{x_\beta : \beta < \omega_1\}$ such that $x_\beta(f^*) = x_\beta^p(f)$ for any $p \in G$ and $\beta < \alpha^p$.

Lemma 7.1. *Suppose that $p \Vdash \dot{f} \in C_I(K^*)$, then there is $f \in C_I(K)$ and $q \leq p$ such that $q \Vdash \dot{f}^* = \dot{f}$.*

Proof. In $V[G]$ there is a sequence of $f_n^* = \sum_{i \in \mathbb{N}} r_{in} \chi_{[a_{in}]}^*$ such that f_n uniformly converges to f where r_{in} s are reals and a_{in} s are elements of $Co(\omega_2)$. Both the reals and the elements of $Co(\omega_2)$ are the same in V and $V[G]$ and so building a decreasing sequence of conditions of \mathbb{Q} we can successively decide them. Find the lower bound q of such a sequence, obtained by 4.6 and define f to be the uniform limit of $f_n = \sum_{i \in \mathbb{N}} r_{in} \chi_{[a_{in}]}$ s (which must exist since the sequence must be a Cauchy sequence as it converges in $V[G]$). Two continuous functions agreeing on a dense set must be equal, so $q \Vdash \dot{f}^* = \dot{f}$. \square

Lemma 7.2. *Suppose that $\mathcal{F} \subseteq C_I(K)$ is countable, then $\nabla \mathcal{F} = \nabla \mathcal{F}^*$, and so $C_I(\nabla \mathcal{F}) = C_I(\nabla \mathcal{F}^*)$.*

Proof. Let \mathcal{A} be a countable subalgebra of $Co(Fr(\omega_2))$ such that f depends on \mathcal{A} for every $f \in \mathcal{F}$. Let $u \in K^*$ be such that $(\Pi \mathcal{F}^*)(u) = t \in [0, 1]^{\mathcal{F}^*}$. $u \cap \mathcal{A}$ belongs to V as it is a countable subset of V . Now extend u to any ultrafilter v of $Co(Fr(\omega_2))$ which is in V , we have $(\Pi \mathcal{F})(v)(f) = (\Pi \mathcal{F}^*)(u)(f^*)$. \square

Lemma 7.3. *Suppose that $p \in \mathbb{Q}$ and $p \Vdash \dot{f} \in C_I(\nabla \mathcal{F}_G^*)$. There is $q \leq p$ in \mathbb{Q} and $g \in C_I(\nabla \mathcal{F}_q)$ such that*

$$q \Vdash \check{g} \circ \pi_{\mathcal{F}_q^*, \mathcal{F}_G^*} = \dot{f}$$

Proof. As in 2.10 one can prove that f depends on countably many coordinates in \mathcal{F}_G^* . One can decide this countable set. Using the compatibility of all elements in the generic G and the fact that \mathbb{Q} is σ -closed one can built $q' \leq p$ such that

$$q' \Vdash \dot{f} = \dot{g} \circ \pi_{\mathcal{F}_{q'}^*, \mathcal{F}_G^*}, \quad \dot{g} \in C_I(\nabla \mathcal{F}_{q'}^*).$$

So now find $q \leq q'$ which decides \dot{g} as g using the previous lemma. \square

8. THE CONSTRUCTION AND THE PROPERTIES OF THE SPACE

Lemma 8.1. *The compact space L with its subset \mathcal{X} has the following properties in $V[G]$:*

- (A) *The weight of L is ω_2 .*
- (B) *$\mathcal{X} = \{x_\beta : \beta < \omega_1\}$ is a dense subset of L*
- (C) *Given*
 - (a) *a sequence of pairwise disjoint elements $(f_n : n \in \mathbb{N})$ of $C_I(L)$,*
 - (b) *a sequence $(\xi_m : m \in \mathbb{N}) \subseteq \omega_1$ such that $f_n(x_{\xi_m}) = 0$ for all $n, m \in \mathbb{N}$, and $\{x_{\xi_m} : m \in \mathbb{N}\}$ is relatively discrete,**there is an infinite $A' \subseteq \mathbb{N}$ such that*
 - (1) *The supremum $f = \sup\{f_n : n \in A'\}$ exists in $C_I(L)$,*
 - (2) *$\overline{\{x_{\xi_m} : m \in A'\}} \cap \overline{\{x_{\xi_m} : m \notin A'\}} \neq \emptyset$*
- (D) *If U_1, U_2 are two open subsets of L then $\overline{U_1} \cap \overline{U_2} = \emptyset$ or $\overline{U_1} \cap \overline{U_2}$ has at least two points.*

PROOF: (A) follows from 2.8, 3.2 (2) and the fact that $\bigcup\{A_p : p \in G\}$ contains unbounded in ω_2 set of limit ordinals by the standard density argument, which can be easily obtained from 4.3. (B) follows from the standard density arguments since we can increase α_p s arbitrarily in ω_1 .

To get (C) work in V and fix $p \in \mathbb{Q}$. Let \dot{f}_n 's, \dot{x}_{ξ_m} 's, be \mathbb{Q} -names for the objects mentioned in items a) - b) of (C). We will produce $q \leq p$ which will force (1) - (2) of the lemma. By 7.3 and 4.6 there is $q' \leq p$ and ξ_m s and functions $g_n : \nabla \mathcal{F}_{q'} \rightarrow [0, 1]$ such that $\xi_m < \alpha_{q'}$, $q' \Vdash \check{\xi}_m = \xi_m$, and such that

$$q' \Vdash \check{g}_n \circ \pi_{\mathcal{F}_{q'}^*, \mathcal{F}_G^*} = \dot{f}_n.$$

It follows that g_n s are pairwise disjoint and that $g_n(x_{\xi_m}^{q'}) = 0$ for all $m, n \in \mathbb{N}$.

Now use 6.3 to find an extension $q \leq q'$ where we have an indestructible supremum of g_n 's for $n \in A'$ and a promise that $\overline{\{x_{\xi_m} : m \in A'\}} \cap \overline{\{x_{\xi_m} : m \notin A'\}} \neq \emptyset$. The supremum and the above nonempty intersection of the closures remain in $\nabla \mathcal{F}_G^* = L$ because if they failed this would be witnessed by some countable $\mathcal{F} \subseteq \mathcal{F}_G^*$ (as continuous functions depend on countably many coordinates and separations of closed sets can be obtained by two unions with disjoint closures of finitely many basic open sets) and we could use the σ -closure of \mathbb{Q} and the compatibility of conditions in the generic G to obtain $s \leq q$ such that the supremum or the condition about the closures are destroyed in $\nabla \mathcal{F}_s$ which is impossible since the supremum is indestructible and the promise preserved by the definition 4.1 of the order in \mathbb{Q} .

To prove (D), first note that L is c.c.c. as a continuous image of a c.c.c. space $K = Co(\omega_2)$, so we may assume that U_1 and U_2 are unions of countably many basic sets. Deciding them all and using 3.5 we may assume that they are all determined by coordinates of $\nabla\mathcal{F}_p$ for some $p \in \mathbb{Q}$. I.e. that we have V_1, V_2 open subsets of $\nabla\mathcal{F}_p$ such that $U_i = \pi_{\mathcal{F}_p, \mathcal{F}}^{-1}[V_i]$ for $i = 1, 2$, where \mathbb{Q} forces that $\dot{\mathcal{F}} = \{\mathcal{F}_p : p \in G\}$. Of course we must have $\overline{V}_1 \cap \overline{V}_2 \neq \emptyset$. Now find an $A \subseteq \omega_2$ such that $A_p < A$ and there is an order preserving bijection $\tau : A_p \rightarrow A$. Transport all the structure of p to A by τ obtaining a condition $p' \in \mathbb{Q}$ such that τ witnesses that p and p' are isomorphic. Amalgamate them according to 3.4 obtaining $q \leq p, p'$. Using 2.5 it is easy to prove that

$$\nabla F_q = (\nabla F_p) \times (\nabla F_{p'}).$$

So taking two distinct points $x, y \in \nabla F_{p'}$ we have that $(t, x) \in \overline{V}_1 \times \{x\} \cap \overline{V}_2 \times \{x\} \subseteq \overline{U}_1 \cap \overline{U}_2$ and $(t, y) \in \overline{V}_1 \times \{y\} \cap \overline{V}_2 \times \{y\} \subseteq \overline{U}_1 \cap \overline{U}_2$ for any $t \in \overline{V}_1 \cap \overline{V}_2$. So, $(t, x), (t, y) \in \overline{U}_1 \cap \overline{U}_2$ completing the proof of (C).

Theorem 8.2. *L is a compact space such that*

- (1) *The density of $C(K)$ is $2^{2^\omega} > 2^\omega$.*
- (2) *Every linear bounded operator $T : C(L) \rightarrow C(L)$ is of the form $T(f) = gf + S(f)$ where $g \in C(K)$ and S is weakly compact (strictly singular)*
- (3) *$C(K)$ is an indecomposable Banach space, in particular it has no complemented subspaces of density less or equal to 2^ω ,*
- (4) *$C(K)$ is not isomorphic to any of its proper subspaces nor any of its proper quotients.*

Proof. (1) follows from (A) of 8.1, a standard counting argument in a generic extension (see [13]) and the fact that \mathbb{Q} does not add new reals to a model of CH. (2) is proved the following way: first we prove that every operator on $C(L)$ is a weak multiplier (see of 2.1 and 2.2. of [10]) exactly the same way as Lemma 5.2 of [10], then we use the fact that for every $x \in L$ the space $L \setminus \{x\}$ is C^* -embedded in L which follows from (C) of 8.1 and Lemma 2.8. of [10]. Now Lemma 2.7 of [10] implies (2) of our theorem. (3) follows, for example, from Lemma 3.4. of [4]. (4) follows from 2.3 of [10] and the fact that of course, L cannot have a nontrivial convergent sequence which would give a complemented copy of c_0 contradicting (3). \square

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